

THE MATHEMATICAL GAZETTE

EDITED BY

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DE ARTE SUPPUTANDI.

TUNSTALL's book is the first work on arithmetic to be printed in England, the first edition of 1522 having its title within a woodcut border, with lively figures by Hans Holbein.

De Morgan had a high opinion of its merits. He writes: "This book is decidedly the most classical which was ever written on the subject in Latin, both in purity of style and goodness of matter. . . . For plain common sense, well expressed, Tunstall's book has been rarely surpassed, and never in the subject of which it treats."

Cuthbert Tunstall studied at Balliol, King's Hall (afterwards incorporated in Trinity), and Padua. His interests were wide, and he was on friendly terms with several of the great renaissance scholars, such as Thomas More, to whom this book is dedicated. At the time, More was Under-Treasurer and Tunstall says that the book could not be dedicated to anybody more appropriately than to one altogether occupied in calculation in the royal treasury. Tunstall had already made some mark in diplomacy, and promotion in the church was rapid: Bishop of London in 1522, he was translated to the much wealthier Prince-Bishopric of Durham in 1529. This year saw the fall of Wolsey and the first steps of Henry VIII towards a breach with Rome. Tunstall, though a man of the "new learning", was of the old faith; under Edward VI his bishopric was suppressed, to be restored under Mary, but since he could not bring himself to support the Elizabethan settlement, he was deprived of it in 1559, and died later that year at the age of 85. He can claim, not indeed the fame of a martyr, but the higher fame, for that age, of never making martyrs.

For the plate of the title-page here reproduced, we are indebted to the Photographic Department of the British Museum.

ABSTRACT ANALYSIS.*

By F. SMITHIES.

The notion of limit.

THE operations of elementary algebra are of finite character; that is to say, an algebraic expression depends, explicitly or implicitly, only on a finite number of variables. The step from algebra to analysis is taken when we begin to consider expressions depending on an infinite number of variables; for instance, the limit of a sequence (a_n) depends on the infinite set of variables a_1, a_2, \dots . Such an infinite set of variables is most conveniently thought of as the set of values of a function; thus the sequence (a_n) is a function of the positive integral variable n , the single term a_n , for example, being the value of the function when n is given the particular value 3. Thus the characteristic features of analysis are (i) the systematic use of functions of a variable capable of an infinity of distinct values, and (ii) the consideration of expressions that involve the values of such a function for an infinity of values of the independent variable.

The simplest expression of this kind is provided by the notion of limit. We meet this first in two distinct forms; the limit of a sequence and the limit of a function of a real variable. The formal definitions are as follows.

The sequence (a_n) is said to tend to the limit l as n tends to infinity if, corresponding to every positive real number ϵ , there is a positive integer N such that $|a_n - l| < \epsilon$ whenever $n > N$. We then write

$$\lim_{n \rightarrow \infty} a_n = l.$$

The function $f(x)$ of the real variable x is said to tend to the limit l as x tends to a if, corresponding to every positive real number ϵ , there is a positive real number δ such that $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$. We then write

$$\lim_{x \rightarrow a} f(x) = l.$$

There are several variations on the second definition. We may be interested, for example, in the limit of $f(x)$ as $x \rightarrow \infty$ or in the "one-sided" limit in which x is always greater than a (usually denoted by $\lim_{x \rightarrow a+0} f(x)$). Sometimes it is more convenient to omit the condition that $|x - a| > 0$; the statement that $\lim_{x \rightarrow a} f(x) = l$ then implies that $f(a) = l$.

If these were the only types of limit that we ever had to deal with, an abstract analysis of the concept of limit would not be worth while. However, this is not the case; the development of mathematics has led to the introduction of many other types of limit, the most elementary of these being double limits such as

$$\lim_{m, n \rightarrow \infty} a_{mn}, \quad \lim_{x \rightarrow a, y \rightarrow b} f(x, y).$$

Limits of functions of several variables.

A convenient way of thinking of a function $f(x_1, x_2, \dots, x_n)$ of n real variables is to regard it as a function defined in the n -dimensional Euclidean space R^n . A point x of R^n is defined as an ordered set of n real numbers x_1, x_2, \dots, x_n ; we write

$$x = (x_1, x_2, \dots, x_n).$$

* A paper given at the Annual Meeting of the Mathematical Association, January 1950.

We define the *distance* $d(x, y)$ between two points x and y by the equation

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2},$$

the natural generalisation of the formula obtained in two- or three-dimensional space by using the theorem of Pythagoras.

If A is any set of points in R^n , the point x_0 is said to be *adherent* to A or to belong to the *closure* \bar{A} of A if, given any positive real number ϵ , there is a point x of A such that $d(x, x_0) < \epsilon$. Clearly every point of A belongs to \bar{A} ; \bar{A} consists in fact of A itself together with all its limit-points.

A real-valued function $f(x_1, x_2, \dots, x_n)$ may now be regarded as a function $f(x)$ of a variable point x of R^n . If $f(x)$ is defined in some subset A of R^n , and x_0 is a point of \bar{A} (not necessarily in A), we shall say that $f(x)$ tends to l as x tends to x_0 in A if, corresponding to any positive real number ϵ , there is a positive real number δ such that $|f(x) - l| < \epsilon$ whenever x belongs to A and $d(x, x_0) < \delta$.

We may also consider functions $f(x_1, x_2, \dots, x_n) = f(x)$ that, instead of being real-valued, take values in a Euclidean space R^m of any number of dimensions. In this case the inequality $|f(x) - l| < \epsilon$ has to be replaced by

$$d(f(x), l) < \epsilon.$$

Metric spaces.

The kinds of limit so far discussed usually suffice for elementary purposes. In more advanced work, however, one meets notions like that of a *functional*, which is, roughly speaking, a function whose domain of definition cannot be regarded as a set of points in a finite-dimensional Euclidean space; for instance, the length of a curve is a functional defined for all rectifiable curves, and an expression such as

$$\int_a^b F(x, y, y') dx,$$

of the kind met with in the calculus of variations, is a functional defined for a class of functions $y(x)$. Since a complete curve or a complete function cannot be specified by giving a finite set of real numbers, we are in a genuinely new situation.

If we wish to consider the limit of a functional defined on a certain class of functions, it is natural to proceed by introducing a notion of distance (technically called a *metric*) for these functions, so obtaining a *function space*. Thus the set of all continuous functions $x(t)$ in the closed interval $a \leq t \leq b$ can be made into a function space by defining $d(x, y)$ as the least upper bound of $|x(t) - y(t)|$ as t varies in the interval, i.e.

$$d(x, y) = \sup_{a \leq t \leq b} |x(t) - y(t)|.$$

With this definition the expression

$$I[x] = \int_a^b x(t) dt,$$

for example, is a continuous functional in the space. For, if $x \rightarrow x_0$ in the space, i.e. $d(x, x_0) \rightarrow 0$, it follows from the definition that $x(t) \rightarrow x_0(t)$ uniformly in t , whence $I[x] \rightarrow I[x_0]$.

Function spaces of this kind are special instances of the general notion of *metric space*. A metric space is any collection E of elements x, y, \dots in which there is defined a real-valued function $d(x, y)$ with the properties:

- (a) $d(x, y) \geq 0$ and $d(x, y) = 0$ when and only when $x = y$;
- (b) $d(x, y) = d(y, x)$;
- (c) $d(x, z) \leq d(x, y) + d(y, z)$.

Property (c) expresses the geometrical statement that the sum of two sides of a triangle is greater than or equal to the third side, and is usually known as the *triangle inequality*. These three properties are possessed by all the notions of distance we have so far considered; we therefore regard $d(x, y)$ as a distance function or metric in the space E .

Now let $f(x)$ be a function defined in a certain metric space E , and taking values in a second metric space E' (which might simply be the real number system with the ordinary definition of distance). We shall say that $f(x)$ has the limit l as x approaches x_0 if, given any positive number ϵ , there is a positive number δ such that $d(f(x), l) < \epsilon$ whenever $d(x, x_0) < \delta$. Here again we may restrict x to some subset A of E and x_0 may be any point of A or of its closure \bar{A} .

General topological spaces.

So far our definition is verbally almost identical with the definition for real-valued functions of a real variable. There are cases, however, where it is inconvenient, or even impossible, to introduce a metric that will lead to the kind of limit that we want to consider, and something still more general is required. The kind of generality we need can be seen by formulating the notion of limit in a vague and intuitive way; we want to be able to say that $f(x)$ has the limit l when x approaches x_0 if we can ensure that $f(x)$ is as near l as we please by taking x sufficiently near x_0 . Thus all we need is some precise formulation of what we mean by "nearness" or "neighbourhood".

We begin by giving a precise definition of a neighbourhood in a metric space. The set V in a metric space E is said to be a *neighbourhood* of the point x_0 if there is a positive number δ such that every point x for which $d(x, x_0) < \delta$ belongs to V . In other words, V contains a complete "sphere" with "centre" x_0 .

The definition of limit can now be re-stated in terms of neighbourhoods: the function $f(x)$ has the limit l when x approaches x_0 if, given any neighbourhood V' of l , there is a neighbourhood V of x_0 such that $f(x)$ belongs to V' whenever x belongs to V .

The neighbourhoods of a point in a metric space can easily be shown to have the following five properties:

- (a) every set containing a neighbourhood of x_0 is a neighbourhood of x_0 ;
- (b) the intersection (i.e. the common part) of any finite set of neighbourhoods of x_0 is a neighbourhood of x_0 ;
- (c) x_0 belongs to all its neighbourhoods;
- (d) every neighbourhood V of x_0 contains a neighbourhood W of x_0 such that V is a neighbourhood of every point of W ;
- (e) if x_0 and y_0 are distinct points, there exist neighbourhoods V , W of x_0 , y_0 respectively such that V and W have no points in common.

We now reverse our procedure. Suppose that we can associate with every point x_0 of a set E a system of sets V , which we may call the neighbourhoods of x_0 , possessing the first four of the five properties given above. Then E is said to be a *topological space*, and the systems of sets V define a *topology* on E . In any such space our last formulation of the definition of limit can be retained word for word. A space that satisfies the fifth of the above conditions as well as the first four is called a *Hausdorff space*; the additional condition guarantees the uniqueness of the limit when it exists.

Application to elementary cases.

The definition of limit that we have now reached seems to be sufficiently

general to include almost every kind of limit that mathematicians ever want to use; there are some exceptions, but these are not of fundamental importance.

Let us now see how the definition can be made to include the limit of a sequence (a_n) of real numbers. To do this, we form a topological space E whose elements are the positive integers $1, 2, \dots$, together with an ideal element denoted by ∞ . Any set containing an integer n will be regarded as a neighbourhood of n ; a set will be regarded as a neighbourhood of ∞ if it contains ∞ together with all but a finite number of the positive integers. The axioms (a)-(e) of a Hausdorff space are then satisfied. By our general definition, to say that $a_n \rightarrow l$ as $n \rightarrow \infty$ means that a_n will lie in a given neighbourhood, say $(l - \epsilon, l + \epsilon)$, for all but a finite number of the integers n ; this is plainly equivalent to the elementary definition.

The procedure just described involves a good deal of apparently irrelevant apparatus; in order to define the topological space E , we have had to manufacture neighbourhoods for the integers (and have never used them), and we have had to introduce the ideal element ∞ . It would clearly be worth while to rid ourselves of these complications, and this we shall now proceed to do.

Filters.

The above discussion of convergent sequences suggests that we should concentrate our attention on the properties of the system of neighbourhoods of a single point in a topological space, ignoring all the properties that involve the neighbourhoods of several points simultaneously. In other words, we are really interested only in the first three of the properties that we used to define a topological space; given any system of sets V associated with the point x_0 and possessing these three properties, we are in a position to define $\lim_{x \rightarrow x_0} f(x)$ in the same way as before.

It now becomes apparent that the right place to put the emphasis is on the system of sets V rather than on the point x_0 , and in our next definition the point x_0 disappears from view altogether.

A system \mathcal{F} of sets A, B, C, \dots , all contained in some set E , is said to be a *filter* if it has the following properties:

- I. Any set B containing a set A of \mathcal{F} also belongs to \mathcal{F} . In particular, the set E belongs to \mathcal{F} .
- II. The common part of two sets A and B of \mathcal{F} belongs to \mathcal{F} .
- III. The empty set does not belong to \mathcal{F} .

Properties II and III together imply that the common part of two sets of \mathcal{F} is never empty.

We now say that $f(x) \rightarrow l$ along the filter \mathcal{F} , writing

$$\lim_{\mathcal{F}} f(x) = l,$$

if, given any neighbourhood V' of l , there is a set A of \mathcal{F} such that $f(x)$ belongs to V' for all x of A .

Since the neighbourhoods of a point in a topological space clearly form a filter, the new definition includes the old one as a special case. To apply the new definition to a convergent sequence of real numbers, we proceed as follows: a set A of positive integers is to belong to the filter \mathcal{F} if and only if A contains all but a finite number of the positive integers. Consequently, $\lim_{\mathcal{F}} a_n = l$ means that a_n will lie in a given neighbourhood V' of l for all but a finite number of values of n ; this clearly coincides with the elementary definition.

We have thus arrived at a general definition of limit which includes all the important kinds of limit that turn up in practice, and which is free from the

irrelevant complications that arise if we have to manufacture a topological space whenever we want to discuss a limit. It turns out, too, that the notion of filter, once it has become familiar through use, is a very convenient one and is extremely easy to handle.

One further point may be mentioned here. In teaching elementary analysis one always has to emphasise that a statement such as " $n \rightarrow \infty$ " or " $x \rightarrow x_0$ " is meaningless in isolation. With filters the situation is different. Suppose that the filter consists of sets contained in a topological space E ; then we can say that the filter \mathcal{F} converges to the point x_0 , and write $\mathcal{F} \rightarrow x_0$, if every neighbourhood of x_0 contains a set of \mathcal{F} ; in virtue of condition I, this is equivalent to saying that every neighbourhood of x_0 belongs to \mathcal{F} . In particular, the filter consisting of all the neighbourhoods of x_0 is convergent to x_0 in this sense. We also remark that if $f(x) \rightarrow l$ as $x \rightarrow x_0$, it follows that $\lim_{\mathcal{F}} f(x) = l$ for every filter \mathcal{F} that converges to x_0 . A special case of this result is the fact that if $f(x) \rightarrow l$ as $x \rightarrow x_0$, then $f(x_n) \rightarrow l$ for every sequence (x_n) such that $\lim_{n \rightarrow \infty} x_n = x_0$.

Directed sets and directed families.

We were led to the notion of filters by a process starting from limits of functions of real variables; an alternative treatment starts from limits of sequences.

We first introduce the notion of a partially ordered set. Suppose that in a set E certain pairs of elements are connected by a relation $x \prec y$, which satisfies the following conditions:

- A. For all x in E , $x \prec x$.
- B. If $x \prec y$ and $y \prec z$, then $x \prec z$.
- C. If $x \prec y$ and $y \prec z$, then $x \prec z$.

The set E is then said to be *partially ordered* by the relation \prec . If the condition:

- D. For all x, y in E , either $x \prec y$ or $y \prec x$;

also holds, E is said to be *totally ordered*.

A partially ordered set Δ is called a *directed set* if, given x and y in Δ , there is an element z of Δ such that $x \prec z$ and $y \prec z$. Every totally ordered set is clearly a directed set, but the converse is false.

We now give some examples of directed sets.

(i) The set of positive integers, with the ordering relation \leq , is totally ordered, and therefore directed.

(ii) The class of all subsets of a given set E , with the ordering relation " A contains B ", is partially ordered; since A and B both contain their common part, it is a directed set.

(iii) The set of all partitions $\Pi(a = x_0 < x_1 < \dots < x_n = b)$ of an interval (a, b) into a finite number of sub-intervals, with the ordering relation " Π_1 gives rise to Π_2 by the adjunction of additional points of subdivision", i.e. " Π_2 is a refinement of Π_1 ", is partially ordered. Since any two partitions have a common refinement, this is also a directed set.

Many other examples could be given.

If to each element α of a directed set Δ is made to correspond an element x_α of some set E , the correspondence is said to define the *directed family* (x_α) . We think of the elements α, β, \dots of Δ as indexing the elements x_α, x_β, \dots of E . Example (i) shows that a directed family may be regarded as a generalisation of a sequence.

We are now in a position to define the limit of a directed family (x_α) whose elements all belong to some topological space, e.g. the real number system. We say that the directed family (x_α) converges to the limit l if, given any

neighbourhood V' of l , there is an element α_0 of Δ such that α_x belongs to V' whenever $\alpha_0 \prec \alpha$.

The close resemblance between this definition and that of the limit of a sequence is obvious. What is more surprising at first sight is that the theory of directed sets and families is completely equivalent to the theory of filters; any notion that can be expressed in terms of the one can also be expressed in terms of the other. Roughly speaking, to get from filters to directed sets we remark that every filter is a directed set, ordered by the relation " Δ contains B ". There are some further technical details, but these we omit. To make the transition in the opposite direction, we define $R(\alpha_0)$ to be the set of all elements α of the directed set Δ such that $\alpha_0 \prec \alpha$, and consider the filter \mathfrak{F} consisting of all the subsets of Δ that contain some $R(\alpha_0)$.

Whether one uses filters or directed sets is a matter of convenience and individual preference; in some contexts filters work more smoothly, in others one finds directed sets easier to handle. One illustration of the use of directed sets is connected with example (iii) above; the upper and lower Riemann sums of a bounded function $f(x)$, regarded as functions of the partition Π , form two directed families. We then define the function $f(x)$ to be integrable if these two families converge to the same limit, and go on to develop the whole theory of the Riemann integral on this basis.

Historical remarks.

The theory of filters was originated by H. Cartan [3], and a full account of its application is given by N. Bourbaki [2]. Directed sets were first introduced by E. H. Moore [5], and the theory was further developed by Moore and H. L. Smith [6]; the convergence of a directed family is sometimes described as *Moore-Smith convergence*. Their applications in general topology have been exploited by G. Birkhoff [1], J. W. Tukey [7] and others. Reference may also be made to the account given by S. Lefschetz [4].

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F. S.

GLEANINGS FAR AND NEAR.

1656. Speaking of his Parliamentary duties, Mr. Morrison said he had a shop to look after—a talking shop. "I am studying it with great care at the present moment. I knew the last House of Commons very well. In fact I knew it inside out. I could go in and tell whether it was in a good temper or not. Its temper would be affected by human considerations, what the Members had had for breakfast, the weather, and I have a theory that the temper of the House of Commons was affected by the decimal point in the number of Members who had had a row with their wives before they had left home."—*The Times*, March 11, 1950. [Per Mr. S. Rushton.]

THE NETS OF THE REGULAR STAR-FACED AND STAR-POINTED POLYHEDRA.

By C. HOPE.

THE four star polyhedra discussed on pp. 171-173 of Lines' *Solid Geometry* form an interesting series of constructions based on the angle of 72° . The finished products form fine decorations for the mathematics laboratory, as well as visual aids.

Fig. 1a. *The Great Dodecahedron* (Lines, p. 171).

The triangles are 72° , 72° , 36° . Lines marked *C* should be cut through. Broken and unbroken lines should be scored on opposite sides of the paper.

Fig. 1b. *The Great Stellated Dodecahedron* (Lines, p. 172).

The angles are 72° and 36° . The lines marked "cut" should be cut through. Corresponding edges should be joined together. Broken and unbroken lines should be scored on opposite sides of the paper.

Fig. 2a. *The Small Stellated Dodecahedron* (Lines, p. 173).

Broken and unbroken lines should be scored on opposite sides of the paper. The angles are 36° and 108° . The star points are made, in the way one puts the dents in a scout hat, by squeezing the three star-shaped figures.

Another solid may be made from the same net by bringing the tetrahedral indentations of the small stellated dodecahedron into relief. This makes a solid which is an icosahedron with pyramids on each face, the faces of the solid so formed lie in fives on the planes of the icosahedron.

Fig. 2b. *The Great Icosahedron* (Lines, p. 173).

BC is the side of a pentagon.

$AB = LA = CD = MD = MC$ = the length of the diagonal of the pentagon.

GFE is an arc of radius *BE*, centre *B*, cutting *AM* in *G* or *F*. The net is drawn by making the large triangles similar to *BFA* and the small ones similar to *BEC*. The substitution of *G* for *F* gives rise to another solid (by drawing triangles similar to *BGA*) which is a dodecahedron having "fluted pyramids" on its faces. The mensuration of the figure 2b (i) is easy. (Note that angle *GBA* is not 60° .)

Method of Construction.

I have found that the quickest, neatest and least wearing on the nerves method of joining up is by using brown paper adhesive tape, which gives a strong job whilst obviating flaps.

The illustrations of the completed models may also be found in Rouse Ball.

C. HOPE.

1857. The N.U.R. contends that 11 per cent. of all railwaymen get the 92s. 6d. minimum and 55 per cent. less than £5. But British Railways claim that average earnings of all their employees are more than £5 a week. They say that earnings average £5 12s. 7d. a week.—*News Chronicle*, January 3, 1950. Quoted, doubtless, as contradictories. [Per Mr. S. Rushton.]

1858. United States women in 1946 bought 8,000 miles of lipstick, 750,000 boxes of face-powder, and 400,000,000 rouge compacts.—*Sunday Express*, 11th May, 1947. [Per Mr. J. T. Pye.]

1859. Einstein's theory of relativity, first propounded in 1905 and since regarded as proved by the atom bomb, is to be given a further test by a group of American scientists.—*Evening Standard*, May 15, 1947. [Per Mr. L. R. B. Elton.]

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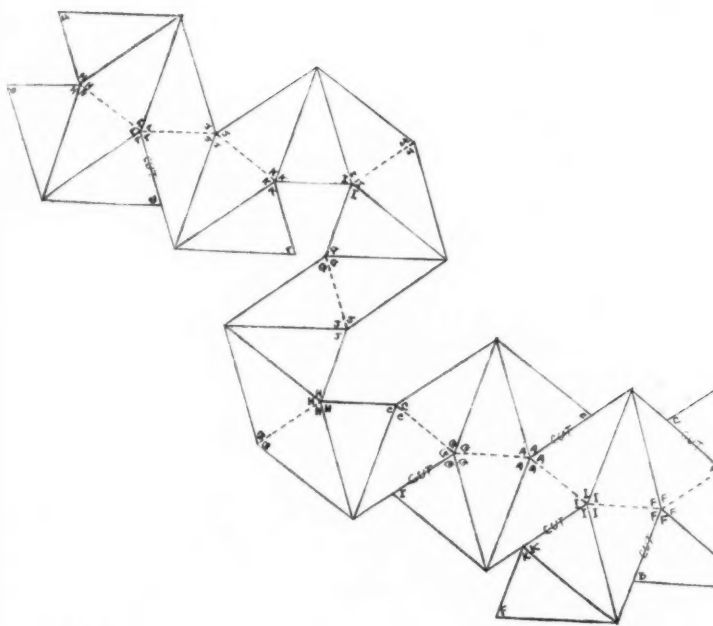


FIG. 10

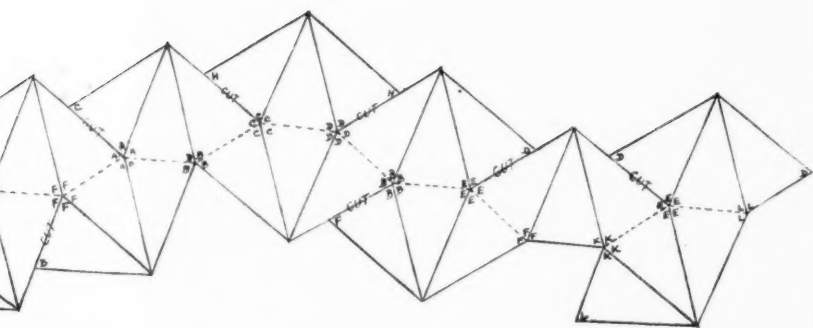


FIG. 1(b).

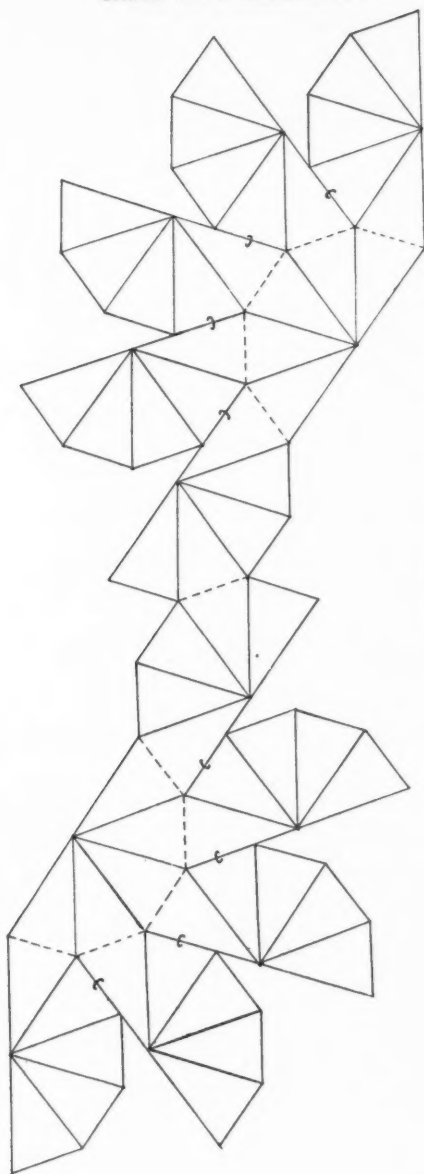


FIG. 1(a)

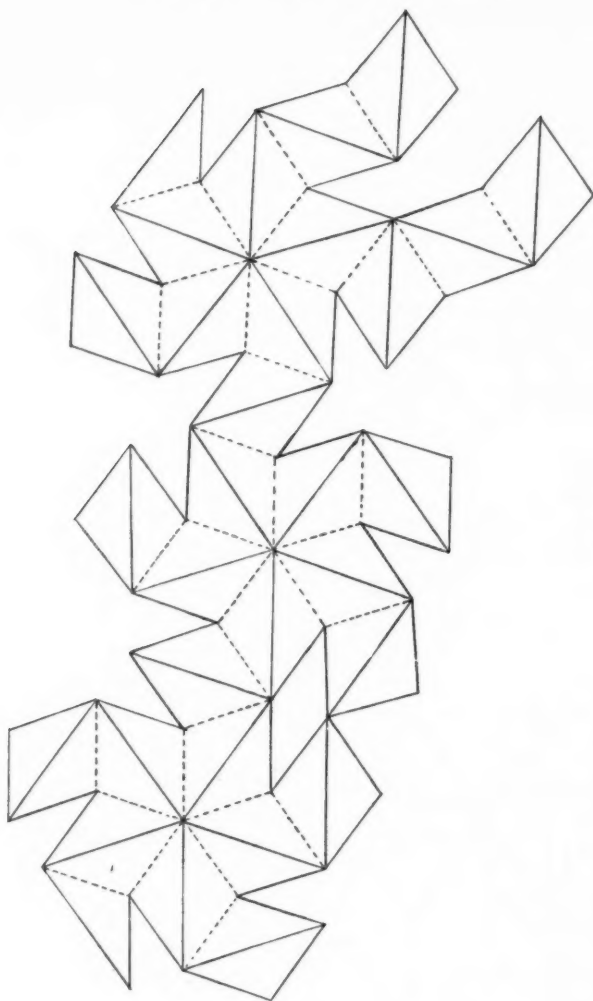


FIG. 2(a)

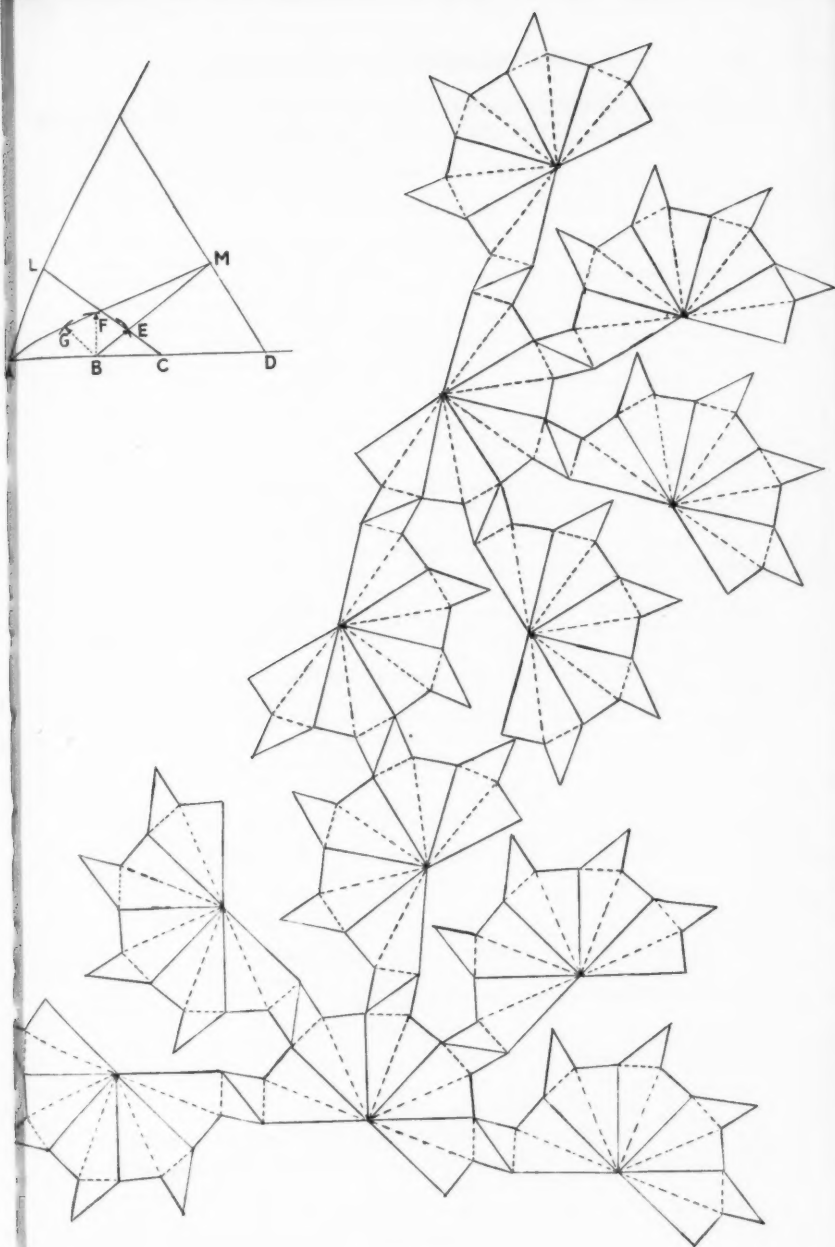


FIG. 2(b)

AN APPLICATION OF CORNU'S SPIRAL TO THE MATHEMATICAL THEORY OF THE MOTION OF AN UNROTATED ROCKET.

BY E. T. J. DAVIES AND V. MAURANEN

1. Introduction

THE mathematical theory of rocket motion has been given by Rosser, Newton and Gross (1947) for the case of unrotated, and slowly rotated, rockets. A rigorous treatment of the subject has been given also by Rankin (1949) in a recent paper which is applicable to both unrotated and rotated rockets.

One of the main objects of such work is the derivation of formulae which may be used to predict the behaviour of the rocket under the action of various disturbing influences. Thus, an angular deviation of the rocket from its normal trajectory may be caused by the thrust not passing through the centre of gravity, by the action of a crosswind (usually assumed constant in the theory), or by the rocket being launched with an initial yaw or initial angular velocity about an axis at right angles to the rocket axis. Malaligned fins may also cause such a deviation.

The exact treatment of such effects is very complicated, but results which are suitable for practical purposes may be obtained when a number of assumptions are made. These include assuming that during the burning period the rocket acceleration and moment of inertia are constant and that the aerodynamic moment is proportional to the square of the velocity. In the case of unrotated rockets, the deviations are then easily obtained when certain "rocket functions" have been calculated. The manner in which such functions enter into the simplified theory may be seen, for the case of constant crosswind, from an article by Knight (1948).

The following work describes a simple graphical method, based on Cornu's spiral, of obtaining the rocket functions which are needed to give the angular deviations at the end of burning due to constant crosswind, initial yaw, initial angular velocity, malaligned thrust, or malaligned exit-plane centre.

The principal rocket functions, required in practice, are

$$E(v_0, v) = \int_{v_0}^v \sin \frac{1}{2}\pi(u^2 - v_0^2) du - \{1 - \cos \frac{1}{2}\pi(v^2 - v_0^2)\} / \pi v, \quad \dots\dots(1.1)$$

$$F(v_0, v) = \frac{1}{2}\pi \left\{ \left[\int_{v_0}^v \cos \frac{1}{2}\pi u^2 du \right]^2 + \left[\int_{v_0}^v \sin \frac{1}{2}\pi u^2 du \right]^2 \right\} + \int_{v_0}^v \sin \frac{1}{2}\pi(u^2 - v_0^2) du, \quad \dots\dots(1.2)$$

$$\text{and} \quad G(v_0, v) = \int_{v_0}^v \cos \frac{1}{2}\pi(u^2 - v_0^2) du - \sin \frac{1}{2}\pi(v^2 - v_0^2) / \pi v. \quad \dots\dots(1.3)$$

The dimensionless parameter v is defined by

$$v = V\sqrt{(n/\pi f)},$$

where

f = rocket acceleration,

V = rocket velocity,

n^2 = an aerodynamic parameter

= (aerodynamic moment at velocity V and yaw δ) / $A V^2 \delta$,

and

A = moment of inertia of the rocket about a transverse axis through the centre of gravity.

Suffices 0 and 1 refer to conditions at launch and at the end of burning respectively, so that at the latter position the rocket functions have the

values $E(v_0, v_1)$, $F(v_0, v_1)$ and $G(v_0, v_1)$. These are the values which are usually required since the subsequent flight of the rocket is largely dependent on the conditions at the end of burning.

The connection between the rocket functions and the corresponding angular deviations may be illustrated briefly for the case of wind. The angular deviation at the end of burning caused by a constant crosswind is

$$\theta = w\sqrt{(n\pi/f)} \cdot E(v_0, v_1).$$

In the case of launch at zero velocity, $v_0 = 0$, and the result is

$$\theta = w\sqrt{(n\pi/f)} \left\{ \int_0^{v_1} \sin \frac{1}{2}\pi u^2 du - [1 - \cos \frac{1}{2}\pi v_1^2]/\pi v_1 \right\},$$

which is equivalent to the formula given by Knight (1948) (see the equation for θ on p. 193 of this reference which contains a small copying error).

2. Approximate values for $E(v_0, v_1)$, $F(v_0, v_1)$ and $G(v_0, v_1)$

A method of calculating the rocket functions has been given by Rankin (1949), based on the use of two functions $A(v)$ and $B(v)$, which appear to have been first introduced by Miller and Gordon (1931). In terms of these functions, the two Fresnel integrals are

$$\int_0^v \cos \frac{1}{2}\pi u^2 du = 1/2 + A(v) \sin \frac{1}{2}\pi v^2 - B(v) \cos \frac{1}{2}\pi v^2, \dots\dots\dots(2.1)$$

$$\int_0^v \sin \frac{1}{2}\pi u^2 du = 1/2 - A(v) \cos \frac{1}{2}\pi v^2 - B(v) \sin \frac{1}{2}\pi v^2. \dots\dots\dots(2.2)$$

It may be shown from equations (2.1) and (2.2) (Preston, 1912, for example) that

$$\pi\sqrt{2} \cdot A(v) = \int_0^\infty \exp[-\pi v^2 t/2] \cdot \frac{t^{-1/2} dt}{1+t^2},$$

$$\pi\sqrt{2} \cdot B(v) = \int_0^\infty \exp[-\pi v^2 t/2] \cdot \frac{t^{1/2} dt}{1+t^2}.$$

The rocket functions can now be written in terms of $A(v)$, $B(v)$, using equations (2.1) and (2.2). It is convenient, however, to introduce three other functions, defined by

$$A_1(v) = 1/\pi v - A(v),$$

$$a(v_0, v) = A(v_0)B(v) + B(v_0)A_1(v),$$

$$b(v_0, v) = B(v_0)B(v) - A(v_0)A_1(v).$$

Equations (1.1), (1.2) and (1.3) may then be written :

$$E(v_0, v) = A(v_0) + A_1(v) \cos \frac{1}{2}\pi(v^2 - v_0^2) - B(v) \sin \frac{1}{2}\pi(v^2 - v_0^2) - 1/\pi v, \quad (2.3)$$

$$F(v_0, v) = \frac{1}{2}\pi[A^2(v_0) + B^2(v_0) + A^2(v) + B^2(v)] - \pi b(v_0, v) \cos \frac{1}{2}\pi(v^2 - v_0^2) - \pi a(v_0, v) \sin \frac{1}{2}\pi(v^2 - v_0^2) - A(v)/v, \dots\dots\dots(2.4)$$

$$G(v_0, v) = B(v_0) - A_1(v) \sin \frac{1}{2}\pi(v^2 - v_0^2) - B(v) \cos \frac{1}{2}\pi(v^2 - v_0^2). \dots\dots\dots(2.5)$$

The advantage of the $A(v)$, $B(v)$ functions is that they are positive, steadily decreasing functions of v , whereas the Fresnel integrals exhibit violent oscillations. Tables of $A(v)$, $B(v)$ to $4D$ have been prepared (Rankin, 1949), and may be used to calculate the rocket functions from equations (2.3), (2.4) and (2.5). The range of the tables covers all likely applications, and the intervals are suitable for linear interpolation.

At the end of burning v reaches its maximum value $v=v_1$, and as $A(v)$, $B(v)$ and $A_1(v)$ all decrease steadily with increasing v , it is clear that the values of $E(v_0, v_1)$, $F(v_0, v_1)$ and $G(v_0, v_1)$ are dependent mainly on the value of v_0 (i.e. on the launching conditions). The function $A(v)$, which is soon approximately $1/\pi v$, does not decrease, however, so rapidly as $B(v)$ and $A_1(v)$. We are therefore led to assume the following approximate expressions for the rocket functions at the end of burning :

$$E(v_0, v_1) \simeq A(v_0) - 1/\pi v, \dots\dots\dots(2.6)$$

$$F(v_0, v_1) \simeq [A^2(v_0) + B^2(v_0) - 1/\pi^2 v_1^2] \pi/2, \dots\dots\dots(2.7)$$

$$\text{and} \quad G(v_0, v_1) \simeq B(v_0). \dots\dots\dots(2.8)$$

In deriving the expression for $F(v_0, v_1)$ from equation (2.4) we have written

$$A(v_1) \simeq 1/\pi v_1.$$

The errors introduced by the use of the above equations are examined in paragraph 4. It is easily shown that the maximum errors are approximately

$$\text{Maximum error in } E(v_0, v_1) \simeq \sqrt{[A_1^2(v_1) + B^2(v_1)]}.$$

$$\text{Maximum error in } F(v_0, v_1) \simeq \frac{1}{2} \pi [A_1^2(v_1) + B^2(v_1) + 2\sqrt{[a^2(v_0, v_1) + b^2(v_0, v_1)]}].$$

$$\text{Maximum error in } G(v_0, v_1) \simeq \sqrt{[A_1^2(v_1) + B^2(v_1)]}.$$

The expressions for the rocket functions given by equations (2.6), (2.7) and (2.8) may now be written in terms of the Fresnel integrals to give :

$$E(v_0, v_1) \simeq \int_{v_0}^{\infty} \sin \frac{1}{2} \pi (u^2 - v_0^2) du - 1/\pi v_1, \dots\dots\dots(2.9)$$

$$F(v_0, v_1) \simeq \left\{ \left[\int_{v_0}^{\infty} \cos \frac{1}{2} \pi u^2 du \right]^2 + \left[\int_{v_0}^{\infty} \sin \frac{1}{2} \pi u^2 du \right]^2 - 1/\pi^2 v_1^2 \right\} \pi/2, \dots\dots(2.10)$$

$$G(v_0, v_1) \simeq \int_{v_0}^{\infty} \cos \frac{1}{2} \pi (u^2 - v_0^2) du. \dots\dots\dots(2.11)$$

3. Application of Cornu's spiral

Cornu's spiral (or clothoid) may be defined in terms of a parameter v by

$$x = \int_0^v \cos \frac{1}{2} \pi u^2 du, \quad y = \int_0^v \sin \frac{1}{2} \pi u^2 du. \dots\dots\dots(3.1)$$

The main properties of the curve which we require are readily obtained from equations (3.1), and are

- (i) The length of the arc, measured from the origin, is $s=v$.
- (ii) The slope of the tangent to the curve is $\psi = \pi v^2/2$.
- (iii) The radius of curvature is $ds/d\psi = 1/\pi v$.

The evolute of the spiral is given by

$$X = x - \sin(\pi v^2/2)/\pi v, \quad Y = y + \cos(\pi v^2/2)/\pi v.$$

Spiral and evolute are shown together in Fig. 1, which also shows the construction which will be needed. $P_0(v_0)$, $P_1(v_1)$ are points on the spiral corresponding to launch and end of burning respectively. C is the final point on both spiral and evolute, with coordinates $(\frac{1}{2}, \frac{1}{2})$, and corresponds to $v = \infty$. E_0 is the point on the evolute corresponding to P_0 , and N_0 is the foot of the perpendicular from C on to P_0E_0 .

At P_0 , write $x = x_0$, $y = y_0$, $\psi = \psi_0$, then the equations of the straight lines P_0E_0 and CN_0 are

$$y - y_0 + (x - x_0) \cot \psi_0 = 0,$$

$$y - \frac{1}{2} - (x - \frac{1}{2}) \tan \psi_0 = 0,$$

respectively.

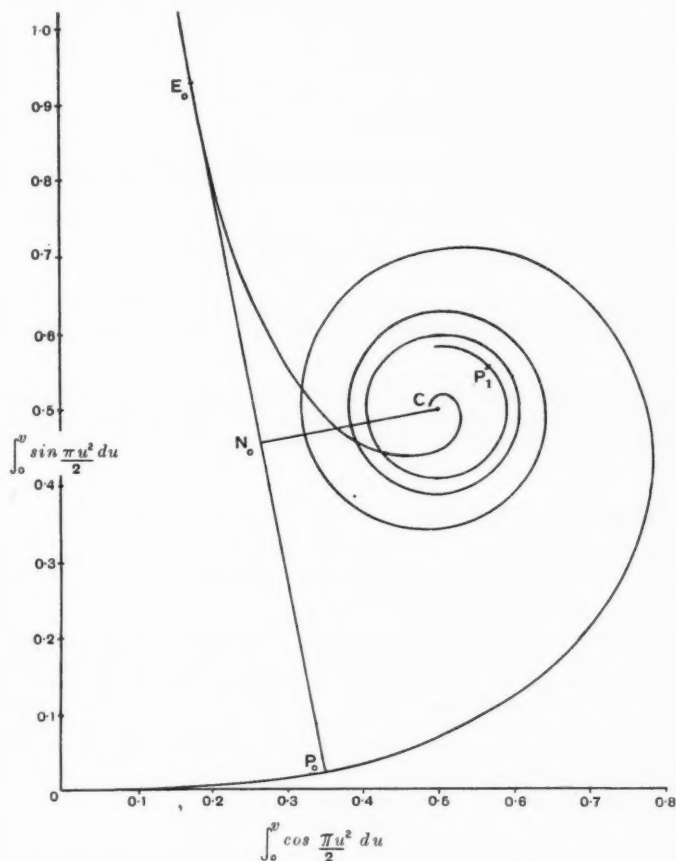


FIG. 1.

It follows that, apart from sign,

$$CN_0 = (\frac{1}{2} - y_0) \sin \psi_0 + (\frac{1}{2} - x_0) \cos \psi_0,$$

and

$$P_0N_0 = (y_0 - \frac{1}{2}) \cos \psi_0 - (x_0 - \frac{1}{2}) \sin \psi_0.$$

But
$$x_0 = \int_0^{v_0} \cos \frac{1}{2} \pi u^2 du, \quad y_0 = \int_0^{v_0} \sin \frac{1}{2} \pi u^2 du, \quad \psi = \pi v_0^2/2,$$

so that
$$CN_0 = \sin \frac{1}{2} \pi v_0^2 \int_{v_0}^{\infty} \sin \frac{1}{2} \pi u^2 du + \cos \frac{1}{2} \pi v_0^2 \int_{v_0}^{\infty} \cos \frac{1}{2} \pi u^2 du,$$

and
$$P_0N_0 = \sin \frac{1}{2} \pi v_0^2 \int_{v_0}^{\infty} \cos \frac{1}{2} \pi u^2 du - \cos \frac{1}{2} \pi v_0^2 \int_{v_0}^{\infty} \sin \frac{1}{2} \pi u^2 du.$$

That is,
$$CN_0 = \int_{v_0}^{\infty} \cos \frac{1}{2} \pi (u^2 - v_0^2) du,$$

and
$$P_0N_0 = \int_{v_0}^{\infty} \sin \frac{1}{2} \pi (u^2 - v_0^2) du,$$

a minus sign being omitted in the last equation, since the integrals are both positive.

It follows from equations (3.1) that

$$(P_0C)^2 = \left[\int_{v_0}^{\infty} \cos \frac{1}{2} \pi u^2 du \right]^2 + \left[\int_{v_0}^{\infty} \sin \frac{1}{2} \pi u^2 du \right]^2.$$

From the radius of curvature property ((iii) above), it follows that if E_1 is a point on the evolute corresponding to the point P_1 on the spiral, then

$$P_1E_1 = 1/\pi v_1.$$

It will be seen from Fig. 1 that the evolute decreases rapidly from large positive values and describes a spiral around C . For comparatively small values of v , the evolute is very close to the point C , so that to a good approximation

$$P_1C \cong 1/\pi v_1. \quad \dots\dots\dots(3.2)$$

The equations (2.9), (2.10) and (2.11) may now be written :

$$E(v_0, v_1) \cong P_0N_0 - P_1C, \quad \dots\dots\dots(3.3)$$

$$F(v_0, v_1) \cong \frac{1}{2} \pi [(P_0C)^2 - (P_1C)^2], \quad \dots\dots\dots(3.4)$$

$$G(v_0, v_1) \cong CN_0. \quad \dots\dots\dots(3.5)$$

A simple graphical method for obtaining the rocket functions at the end of burning is now available. Spiral and evolute are drawn on as large a scale as convenient, and the arcs of both curves are marked off and labelled according to the value of the parameter v . The points P_0 , P_1 and E_0 are then obtained easily, and the construction is simply to join P_0E_0 and draw CN_0 perpendicular to P_0E_0 . Measurement of P_0N_0 , CN_0 , P_0C and P_1C then gives the right hand sides of equations (3.3), (3.4) and (3.5).

4. Accuracy of the approximations

The errors of the method divide into those associated with the mathematical approximations and those involved in drawing and measuring the curves. The latter errors will not be considered here ; it is thought they may be kept as small as required by careful drawing and reproduction on a suitable scale.

The main approximations arise from the use of equations (2.6), (2.7) and (2.8). The maximum percentage errors which result have been calculated for a range of values of v_0 and v_1 , and are given in the following table.

MAXIMUM PERCENTAGE ERRORS

v_1	v_0	0.2	0.5	0.8	1.1	1.4
$E(v_0, v_1)$	2.0	3.8	5.0	7.4	11.8	21.2
	2.5	1.8	2.3	3.3	4.8	7.2
	3.0	1.0	1.3	1.7	2.4	3.4
	3.5	0.6	0.8	1.0	1.4	1.8
	4.0	0.4	0.5	0.7	0.9	1.2
	4.5	0.3	0.3	0.4	0.6	0.8
	5.0	0.2	0.2	0.3	0.4	0.5
$F(v_0, v_1)$	2.0	4.5	6.4	9.3	14.3	24.0
	2.5	2.3	3.2	4.4	6.2	9.0
	3.0	1.3	1.8	2.4	3.4	4.4
	3.5	0.8	1.1	1.5	1.9	2.6
	4.0	0.5	0.8	0.9	1.2	1.5
	4.5	0.4	0.5	0.7	0.9	0.9
	5.0	0.3	0.3	0.6	0.6	0.5
$G(v_0, v_1)$	2.0	3.6	6.9	13.0	23.5	40.5
	2.5	1.9	3.7	6.9	12.5	21.5
	3.0	1.1	2.1	4.0	7.2	12.5
	3.5	0.7	1.3	2.5	4.5	7.8
	4.0	0.5	0.9	1.7	3.1	5.4
	4.5	0.3	0.6	1.2	2.2	3.7
	5.0	0.2	0.5	0.9	1.6	2.7

Some of the values of v_0, v_1 given in the table are unlikely to occur in practice. As already explained, v_1/v_0 is the ratio of velocity at the end of burning to velocity at launch. In firings from the ground this ratio is unlikely to be less than about five, and may easily be as high as ten. In firings from aircraft, the values are lower, due to the added velocity of the aircraft, and the ratio may be as low as three for projection from high speed aircraft.

At the same time, the value of v_0 is closely linked with the method of projection. In firings from the ground, most practical cases are in the range $0.2 < v_0 < 0.5$ for conventional solid-fuel rockets. The higher values of v_0 shown in the tables are only reached in firings from aircraft, the values attained increasing with aircraft speed.

As a result of these factors, the great majority of all practical cases are to be found in parts of the table where, an inspection shows, the errors consequently are small.

The only other approximation made is that of equation (3.2). The percentage error here depends on v_1 alone and is 1.7, 0.8 and 0.4 for v_1 equal to 2.0, 2.5 and 3.0 respectively. For $v_1 = 3.5$, the percentage error is less than 0.1. The effect of this approximation is therefore quite small also.

It is considered that in the majority of practical applications the overall error will not exceed 2%. This is an acceptable error for work of this kind, where the angular deviations are normally small.

As a rough guide, it may be said that the accuracy will be acceptable if

$$v_1 \geq 2.0 + 1.25v_0.$$

It is concluded that the theory, besides giving an interesting application of Cornu's spiral, provides a useful method of evaluating the rocket functions. The graphical method, using the spiral and evolute, is essentially simple in practice, and should prove of value to those engaged on rocket research and development work.

The notation used above is that familiar to British readers. It may be of interest to note that the functions $rr(w)$ and $rj(w)$ introduced by Rosser, Newton and Gross (1947) are given simply by

$$w = \pi v^2/2, \quad rr(w) = \sqrt{2\pi} \cdot A(v), \quad rj(w) = \sqrt{2\pi} \cdot B(v).$$

ACKNOWLEDGEMENTS

The authors wish to acknowledge the generosity of the Royal Swedish Air Board in facilitating the interchange of material which has resulted in their writing the present paper. In addition, they are indebted to Mr. Anderberg and to Mr. Hancock for the interest they have shown in the work, and to the Chief Scientist, Ministry of Supply, and to the Royal Swedish Air Board for permission to publish.

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V. M.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, i.e. volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

1860. I remember how a few words from Bertrand Russell opened to me the glory of Shakespeare's sonnets.—G. M. Trevelyan, *An Autobiography*, p. 14.

1861. Say one industry employs 10 per cent of all workers, and obtains a 20 per cent increase in money wage-rates. As a consequence of this, prices on the average will rise by 2 per cent. Hence the real wages of 10 per cent of the workers will rise by 18 per cent, but those of the remaining 90 per cent will fall by 2 per cent.—T. Barna, *Profits during and after the War*, Fabian Society, 1945. [Per Mr. J. W. Ashley Smith.]

CENTRE OF CURVATURE FOR THE CONICS.

BY ROBERT C. YATES, West Point, N.Y., U.S.A.

THE determination of centres of curvature and evolutes of well-known curves still forms an item of interest in the customary first course in Calculus. This note calls attention to an elegant construction for the centre of curvature of the conic based upon a property apparently forgotten [1] and offers an excuse for the consideration of evolutes.

1. For a plane curve given in rectangular coordinates

$$|R| = \left| \frac{(1+y'^2)^{\frac{3}{2}}}{y''} \right|, \quad N^2 = y^2(1+y'^2),$$

where R is the radius of curvature and N the length of the normal measured from the curve to the X -axis. Thus for any curve

$$|R| = \left| \frac{N^2}{y^3 y''} \right|.$$

2. Consider the radius of curvature for the general conic

$$y^2 = 2Ax + Bx^2,$$

where A is the semi-latus rectum.

Here

$$yy' = A + Bx, \quad yy'' + y'^2 = B \quad \text{or} \quad y^3 y'' + y^2 y'^2 = By^3.$$

Thus

$$y^3 y'' = By^3 - (A + Bx)^2 = -A^2,$$

and

$$|R| = \left| \frac{N^2}{A^2} \right|,$$

where N is the length of the normal measured from the curve to the focal diameter.

3. In polar coordinates with pole at a focus, the conic is

$$\rho_1 = A/(1 - e \cos \theta).$$

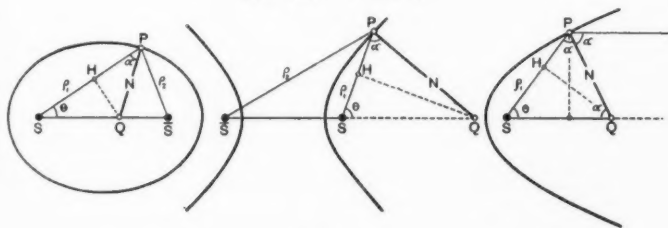


FIG. 1.

Since the normal PQ at any point $P: (\rho_1, \theta)$ bisects the angle between the focal radii, for the central conics

$$(\overline{SQ})/(\overline{SQ}) = \rho_1/\rho_1,$$

or, if unity be added to or subtracted from each member (for ellipse or hyperbola, respectively) :

$$2c/(SQ) = 2a/\rho_1,$$

where a , c have their usual designations and the eccentricity $e = \frac{c}{a}$. That is

$$SQ = e \cdot \rho_1.$$

Now, if H is the foot of the perpendicular from Q upon a focal radius and α the angle this radius makes with the normal,

$$SH = (SQ) \cdot \cos \theta = e \cdot \rho_1 \cdot \cos \theta;$$

and, from the foregoing,

$$PH = \rho_1 - e \cdot \rho_1 \cdot \cos \theta = A = N \cdot \cos \alpha.$$

For the parabola, $e=1$ and $\rho_1 = A + \rho_1 \cdot \cos \theta = SQ$. Thus for all three types

$$PH = N \cdot \cos \alpha = A.$$

Accordingly, for the conics, the projection of the normal length upon a focal radius is constant and equal to the semi-latus rectum.

4. Since $\cos \alpha = \frac{A}{N}$ and $|R| = \left| \frac{N^2}{A^2} \right|$,
there follows

$$|R| = |N| \cdot \sec^2 \alpha.$$

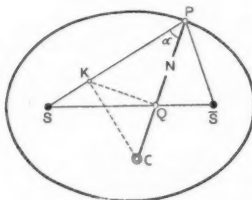


FIG. 2.

Thus to locate C , the centre of curvature, draw the perpendicular to the normal at Q meeting a focal radius in K . The perpendicular at K to this focal radius meets the normal in C .

The point-wise construction of the evolutes of the conics is thus a matter of ruler and compass only. (Fig. 3).

The consideration of evolutes is often a cause of profound wonder on the part of the student. Their appearance as *caustics* * in the theory of physical optics [2] may excuse them as something other than an academic irritation. For example, (Fig. 4), let S be a point source of light whose rays are refracted

* The envelope of rays emitted from a point source after reflection or refraction by a given curve.

at a line L . SQ is incident, QT refracted, and \bar{S} is the reflection of S in L . Produce TQ to meet the circle drawn through S, Q, \bar{S} in P . Let the angles of incidence and refraction be θ_1 and θ_2 and $\mu = \frac{\sin \theta_1}{\sin \theta_2}$ be the index of refraction. SP and $\bar{S}P$ make equal angles with the refracted ray PQT . Thus in passing from a dense to a rare medium: ($\theta_1 < \theta_2$), and *vice versa*: ($\theta_1 > \theta_2$),

$$\mu = \frac{\sin \theta_1}{\sin \theta_2} = \frac{AS}{PS} = \frac{A\bar{S}}{P\bar{S}}.$$

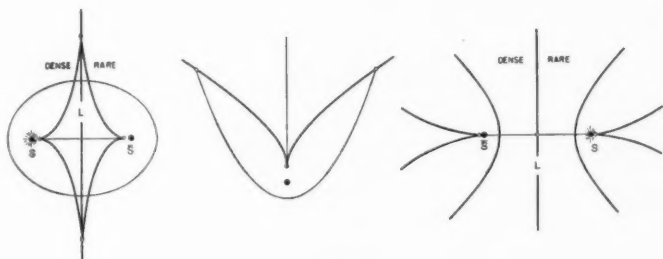


FIG. 3.

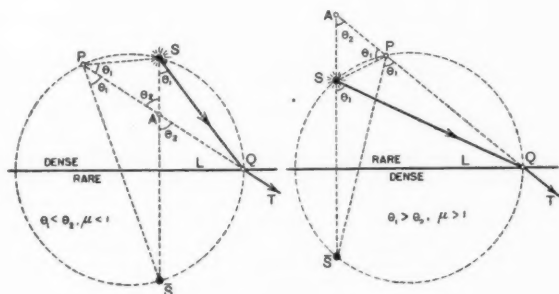


FIG. 4.

$$\mu = \frac{AS + A\bar{S}}{PS + P\bar{S}} = \frac{S\bar{S}}{PS + P\bar{S}}, \mu < 1.$$

Thus, since $S\bar{S}$ is constant,

$$PS + P\bar{S} = S\bar{S}/\mu,$$

a constant. The locus of P then is an ellipse with S, \bar{S} as foci, major axis $S\bar{S}/\mu$, eccentricity μ , and with PQT as its normal. The envelope of these rays PQT , normal to the ellipse, is its evolute, the caustic (Fig. 3).

$$\mu = \frac{A\bar{S} - AS}{P\bar{S} - PS} = \frac{S\bar{S}}{P\bar{S} - PS}, \mu > 1.$$

Thus, since $S\bar{S}$ is constant,

$$P\bar{S} - PS = S\bar{S}/\mu,$$

a constant. The locus of P then is an hyperbola with S, \bar{S} as foci, major axis $S\bar{S}/\mu$, eccentricity μ , and with PQT as its normal. The envelope of these rays PQT , normal to the hyperbola, is its evolute, the caustic (Fig. 3).

These bright curves may be seen on the table when light shines through a glass of water. Other caustics, including the cardioid and nephroid, which have a circle as reflecting curve, can be found on the surface of a finger or napkin ring [3].

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EDINBURGH MATHEMATICAL SOCIETY

ST. ANDREWS MATHEMATICAL COLLOQUIUM 1951

IN prewar years the periodical Mathematical Colloquia held in St. Andrews by the Edinburgh Mathematical Society were deservedly popular with mathematicians from this country and from abroad. The Society is once more organising such a gathering, which will take place in St. Andrews from July 18th to July 28th, 1951. The business of the Colloquium will consist of several short invited courses of lectures on topics of general interest to mathematicians, supplemented by a number of single lectures, which will usually, but not always, be on more specialised subjects.

The Colloquium sub-committee has arranged the following courses:

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Professor J. L. Synge, F.R.S. (Dublin)—“Geometry of Function Space”.

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SOME REMARKS ON THE FOUNDATIONS OF MATHEMATICS.

AN EXPOSITORY ARTICLE.

BY G. KREISEL.

Introduction.

It is tempting to state the problems of the foundations of mathematics in general logical terms, with words like "truth", "proposition", "necessary conclusions". In this way a sometimes useful, and always at any rate verbal, connection is made with old problems of logic; and for this reason also mathematical investigations into the foundations of mathematics have found a good number of popular exponents. But it seems that at times these expositions suffer considerably from rather ill-chosen slogans.

The people who bring their views before the public most easily are the *intuitionists*. They form a school which, roughly speaking, wishes to build up "constructive" mathematics and ignores much of ordinary mathematics, particularly analysis and set theory, on the ground that the latter are either "all wrong" or "do not mean" anything; in particular, *indirect proofs* (*reductio ad absurdum* proofs)* and existence proofs are criticised. The inevitable effect on a practising mathematician is that he applies an indirect argument: if the conclusion of such researches is that most of the mathematics which I do does not mean anything, then there must be a lot wrong with these researches! We shall try to make out below that the *problem* which is at the back of the intuitionist criticism can be formulated quite soberly; but it will not have so startling and general a form as usual.

Another well-known group is Hilbert's school, who are often described as *formalists*, and reputed to have only one interest in mathematics, namely, to establish the *consistency* of formal systems; by that is meant that in the formal systems (of axioms and rules of inference) considered, in which a symbol " \neg " for negation occurs, the formulae \mathcal{A} and $\neg \mathcal{A}$ can not both be proved. Now, the importance of this work is easily missed: for one thing, what one ordinarily sees of formal systems, in books on abstract algebra, is rather artificial stuff, and does not seem to have anything to do with the "natural" arguments of number theory and analysis: hence some of the suspicion of formal systems. Also, whatever logical difficulties one may feel in the development of analysis, say, there is nothing to suggest a plain inconsistency; so the problem of consistency seems a rather pedantic one. We shall try to bring out how the need for a formal system arises quite naturally, and also report on how the consistency proofs provided by the Hilbert school can be made to clear up the intuitionists' problems (made precise in a suitable way). Naturally this applies only to the systems for which such proofs have been given, at present, roughly speaking, analysis (of sequences and continuous functions) with the exclusion of the general theory of sets of points.

The main part of the present article is concerned with the problem of the intuitionists, or, as we prefer to call it, the problem of *interpreting* classical

* The criticism of indirect proofs of theorems without existence symbols is particularly unconvincing; see, for example, *Gazette*, No. 300, pp. 202-3, where it is said that the ordinary proof of the irrationality of $\sqrt{2}$ ($p^2 \neq 2q^2$ for integers p and q) does not "tell" you how much p^2/q^2 differs from 2; yet it tells you that $p^2 \neq 2q^2$ and, since p and q are integers, that $|p^2 - 2q^2| \geq 1$. No attempt is made in this article to describe precisely which proofs (of analysis) are *direct* and which are not. For, we believe, the logical difficulties which are often put down to indirect proofs arise from the very *form* of the theorem which is proved, and our discussion concentrates on this.

theorems. But this is rather an isolated problem in the foundations of mathematics, and it may give the false impression that such an interpretation is a *sine qua non* for the application of mathematics. We shall therefore conclude the article with a brief note on how mere consistency of a formal system may be all we want in an application of the system to physics, or how even an inconsistent system might have a similar use!

The interpretation of theorems in elementary analysis. Arithmetical identities.

It will be most convenient to start with a branch of analysis which may be called *arithmetical identities*, a natural generalisation of school mathematics. The general idea is that we concern ourselves with formulae of the kind :

(some relation) $A(a_1, \dots, a_r)$ holds whatever integers are substituted for the a .

Familiar formulae from ordinary arithmetic, such as

$$a^2 - b^2 = (a + b)(a - b), \quad 2^a \geq a + 1,$$

are examples.

We must explain the words "some relation"; that is, how the formulae $A(a_1, \dots, a_r)$ are built up. The relations which we consider are either

(i) equations between "functions", $f(a_1, \dots, a_r) = g(a_1, \dots, a_r)$, or (ii) inequalities $f(a_1, \dots, a_r) < g(a_1, \dots, a_r)$ or (i) and (ii) joined together by grammatical connections, "and", "or", "if... then", "not"; for example, "if $f(a_1, \dots, a_r) = g(a_1, \dots, a_r)$ then $f_1(a_1, \dots, a_r) < g_1(a_1, \dots, a_r)$ ", for certain f, g, f_1, g_1 .

Finally the functions (of whole numbers) considered here are all systematically computable, that is, defined in such a way that from the definition one can read off their values for any integral argument. Before giving some of the useful schemes for defining such functions, *recursive* definitions of various kinds, we must remind the reader of definitions of functions which are not a systematic technique for the computation of their values; * for example, "the least prime number greater than n " is such a definition of a function of n : from the definition it is not apparent that there is a prime number greater than n ; the fact that there is one less than $n! + 2$ allows the definition to be replaced by a recursive one, but does not make the given definition recursive. More impressive examples are easily got from unsolved problems in arithmetic.

Compare such a definition with *primitive recursive* definitions, which, to distinguish them from other types, may be called of order ω . They are defined by the following scheme :

$a + 1$ is a primitive recursive function with argument a ;

if $g(b)$ and $g_1(a, b, c)$ are such functions,

$f(0, b) = g(b)$, $f(a + 1, b) = g_1(a, f(a, b), b)$ is a primitive recursive definition (of the function $f(a, b)$).

For example, $a + b$, where $g(b) = b$, and $g_1(a, b, c) = b + 1$; similarly $a \cdot b$, 2^a , and even p_n , the n th prime number.

There are other important schemes which may be found in the literature. It is plain from a diagonal argument that there is no one scheme (of a general type) which exhausts all the definitions which one could call a systematic technique of computation.

* In school mathematics such definitions simply do not occur, and the reader might wonder what the fuss is about: hence we give an example.

Limit theorems.

Now consider a limit theorem of analysis, say that a certain monotone increasing sequence of rationals r_n between 0 and 1 converges: to fix ideas suppose that the sequence $r_n = p(n)/q(n)$, is defined by recursive functions $p(n)$, $q(n)$ so that, for any m , there is an n_0 such that for all n ,

$$0 \leq \frac{p(n)}{q(n)} \leq \frac{p(n+1)}{q(n+1)} \leq 1;$$

$$\text{and if } n \geq n_0, \quad \frac{p(n)}{q(n)} - \frac{p(n_0)}{q(n_0)} < \frac{1}{m}.$$

This theorem is of the form, or can easily be brought into the form:

for any m there is an n_0 such that for all n a certain relation $A(m, n_0, n)$ holds, where A is of the form considered in the preceding section.

In fact, it is shown in textbooks on logic, and one can easily convince one's self by going over a book on analysis, that in the part of analysis which we consider, all theorems can be stated in the form:

for all n_1 , there is an m_1 such that for all n_2 there is an $m_2 \dots$ such that for all n_r there is an m_r such that $A(n_1, \dots, n_r, m_1, \dots, m_r)$ holds.(i)

Note. The variables in arithmetical identities are called *free*, the variables $n_1, \dots, n_r, m_1, \dots, m_r$ are called *bound*. Observe also that without serious difficulties *free* function variables can be introduced into arithmetical identities, thus:

$$\{f(a)\}^2 - \{f(b)\}^2 = \{f(a) - f(b)\} \{f(a) + f(b)\}.$$

The problem of interpretation.

Let us consider, quite naively, what arithmetical identities "mean", or, rather, what it means to say that the identities are "true" or "verifiable". Clearly, whatever numbers $(0, 1, 2, \dots)$ are substituted for the number variables (or whatever computable functions for the function variables) the arithmetical identity must reduce to a correct numerical formula; for us, the important point of this observation is that the meaning of the arithmetical identity is largely independent of *how* it was proved, since the notions of correct and incorrect numerical formulae can be defined independently of the system of proofs in which they occur.

But now consider what the limit theorem quoted above means; the naive answer is, of course, that we can find the n_0 , that is, there is a computable definition of a function $N_0(m)$ such that

$$0 \leq \frac{p(n)}{q(n)} \leq \frac{p(n+1)}{q(n+1)} \leq 1,$$

$$\text{and if } n > N_0(m) \text{ then } \frac{p(n)}{q(n)} - \frac{p\{N_0(m)\}}{q\{N_0(m)\}} < \frac{1}{m}.$$

We call this formula a *constructive* version of the original one. (We observe in passing that in the naive answer the bound variable formula is replaced by a free variable formula!) Now the point is that the naive answer is false in a quite precise sense: one can prove that for a certain primitive recursive sequence $p(n)/q(n)$ which is easily shown to be monotone and bounded, there is *no* computable definition of a function $N_0(m)$ as required above. In fact, if one looks at any of the usual proofs of the theorem it is clear that they do nothing towards the definition of such a function $N_0(m)$.

What, then, are we to call the meaning of this formula, or are we to be intuitionists? * Now, if we are modern we may reply that the formula means that it can be proved from certain axioms by certain rules, and the inevitable answer is: so what?, or an interminable discussion on whether or not the rules are "permissible". The discussion is certainly not decided by Hilbert's principle that rules are permissible if they are consistent; for, the absence of a constructive version is no inconsistency in the sense of the introduction: if one is worried about the non-constructive character of a system (which is what we are talking about), why should a proof of the consistency of the rules put things right? But also we do not accept the intuitionist critic of proofs in analysis where only those theorems can be proved which have a constructive version in the sense above, and he rejects the normal system because it does not satisfy his criterion. We do not wish to ignore the theorems of classical analysis, but wish to interpret them.

Formulation of the problem.

The first task is to characterise the vague demand for an interpretation by a *mathematical* problem.

A crucial step in this formulation is a consideration of Hilbert; whenever concepts in mathematics work, but are not "clear", we eliminate them. Complex numbers are replaced by pairs of reals, projection to points at infinity by suitable changes of coordinates, and so forth. He is led to the consistency formulation by a very radical decision: to ignore all bound variable formulae except as methods of proof; he considers only numerical formulae; when they are proved in a system by means of bound variable formulae, the consistency proof provides the elimination of variables from the proof altogether. So to speak, it should show that the methods to which objection has been taken do not produce false results, that is, false *numerical* results.

Since we observed that the trouble over interpretation arises with bound variable formulae we find that the consistency formulation is too narrow. But we keep to the idea of eliminating the bound variables, and argue: *since arithmetical identities have a clear application, and bound variable formulae have not, we wish to give a systematic translation of the latter type into those of the former.*

What properties should such a translation have if it is to be called an interpretation?

Let us denote bound variable formulae of the system considered by German capitals \mathfrak{U} and free variable formulae by Roman capitals A .† Then one might wish to associate with every formulae \mathfrak{U} of the system a formula A so that

if \mathfrak{U} is proved, A is verifiable,

if $\neg \mathfrak{U}$ is proved, we can find a substitution for the variables in A which reduces A to a *false* formula.

Naturally, both the method of finding A , and in case (ii) of finding the substitution from a disproof of \mathfrak{U} should be recursive. Actually it turns out that such a translation is impossible. Instead, one can do this:

* At this point, the false impression mentioned in the introduction may be created: the proofs of a system, that is, the classification of formulae into proved, disproved (and indeterminate) ones, may well be useful without an interpretation. However, if one is determined to have an interpretation, one is led to some such formulation as the one below, I think.

† It is here clear why one needs a formal system. To give precise translation rules, one must at least say *what* formulae are to be translated, and from below, how they are to be proved. Giving a formalization of proofs is just that.

Interpretation.

To any \mathfrak{A} we find a sequence A_n such that

- (α) If \mathfrak{A} is proved we find an n such that A_n is verifiable.
- (β) If $\neg \mathfrak{A}$ is proved for every n we find a substitution for the variables of A_n which reduces A_n to a false formula.

It is convenient to add :

- (γ) If $\mathfrak{A} \rightarrow \mathfrak{B}$ (\mathfrak{A} implies \mathfrak{B}) can be proved in the system, and A_n is verifiable, we find a $B_{\sigma(n)}$ which is also verifiable.

In general, (β) follows from (γ).

For the part of analysis which we consider, such an interpretation can be given. Let \mathfrak{A} be the formula (i) : consider a constructive version of $\neg \mathfrak{A}$, that is, a number N_1 and functions $N_2(m_1)$, $N_3(m_1, m_2)$, \dots , $N_r(m_1, \dots, m_{r-1})$ such that

$$-A[N_1, N_2(m_1), \dots, N_r(m_1, \dots, m_{r-1}), m_1, \dots, m_r]$$

holds whatever numbers are substituted for the m . Actually, if (i) has been proved, for any such number N_1 , and systems of functions $N_{i+1}(a_1, \dots, a_i)$ we can find numbers m_1, \dots, m_r such that the above formula is false ; that is, we have a method of finding a counter-example to any constructive version of $\neg \mathfrak{A}$.

The methods of finding the numbers m in terms of N_1 and $N_{i+1}(a_1, \dots, a_i)$ are recursive *functionals*, that is, definitions whose arguments are the number variable N_1 and the function variables N_{i+1} ; the functionals are such that, for a given N_1 , the value can be calculated from the values of N_{i+1} for a certain finite number of arguments only. (An example of such a functional is the N th iteration used below.) The functionals needed can be ordered, and if the n th set of functionals is $m_1^{(n)}, \dots, m_r^{(n)}$, then

$$A[N_1, N_2(m_1^{(n)}), \dots, N_r(m_1^{(n)}, \dots, m_r^{(n)}), m_1^{(n)}, \dots, m_r^{(n)}]$$

is the formula associated with \mathfrak{A} .

In particular, consider the limit theorem given above. For any m there is an n_0 such that for all n

$$0 \leq \frac{p(n)}{q(n)} \leq \frac{p(n+1)}{q(n+1)} \leq 1,$$

and, if

$$n > n_0, \quad \frac{p(n)}{q(n)} - \frac{p(n_0)}{q(n_0)} < \frac{1}{m},$$

then a constructive version of the opposite, that is, of non-convergence, would be that we should find a number m and some function $N(n_0)$ such that for all n_0

either

$$0 \leq \frac{p\{N(n_0)\}}{q\{N(n_0)\}} \leq \frac{p\{N(n_0)+1\}}{q\{N(n_0)+1\}} \leq 1$$

is false, or

$$N(n_0) > n_0 \quad \text{and} \quad \frac{p\{N(n_0)\}}{q\{N(n_0)\}} - \frac{p(n_0)}{q(n_0)} \geq \frac{1}{m}.$$

This is impossible for all n_0 ; namely, it is false for $n_0 = 0$, or for $n_0 = n_1 = N(0)$, \dots , $n_{i+1} = N(n_i)$, \dots or n_m . (Note that in the statement of the theorem n_0 is an existential variable, in the statement of the opposite n_0 is a universal variable.) Here, the functional used is one of the numbers n_0, \dots, n_m .

Special cases.

In general we cannot find a constructive version of a theorem proved in classical analysis. But here is an exception : if a formula is of the form

for all n there is an m such that a relation $A(n, m)$ holds, then from the proof one can find a recursive function of a certain system (ω^{ω^*}), say $f(n)$, which makes $A\{n, f(n)\}$ an arithmetical identity.

From this can be obtained a useful principle in studying apparently non-constructive proofs of theorems of the simple form above (see, for example, Littlewood, *Gazette*, No. 300, pp. 169-171). If from "for all n , $A(n)$ " follows "for all a , $B(a)$ " and $B(n)$ is false, then we can find an m (or a function $m(n)$) such that $A(m)$: (or $A[m(n)]$) is false. This is used in making Littlewood's proof constructive.

Final note.

The idea of interpreting bound variable formulae by arithmetical identities whose interpretation gives no trouble, is, perhaps, so obvious that one feels: what else could it be? Now, naturally, I feel that it is an appropriate formulation of the problem hinted at by the intuitionist criticism. But it would be a great mistake to think that an interpretation of this kind is necessary to every application of such a formalism. Consider, for example, physics, where with the empirical propositions of a certain branch of the subject are associated formulae of a formal system; this system is then used to pick out true empirical propositions, namely those with which are associated proved formulae of the system; for instance, in the quantum theory, with certain observables, characterised by their classical analogue, are associated differential equations, and the observable can have values λ , if λ is an eigenvalue of the differential equation, and no others. Now consider the (complicated) formulation of " λ is the least eigenvalue of the differential equation $L(D, \mu)$ ". It is: there is a function $\phi(x)$ such that, for all μ , $\phi(x)$ and N , there is an M such that ϕ satisfies the differential equation $L(D, \lambda)$, and has a finite integral ϕ^2 , and for $\mu < \lambda$, if ψ is a solution of $L(D, \mu)$

$$\int_{-M}^M |\psi(x)|^2 dx > N.$$

(Note that a formulation of "is integrable" would introduce a great deal more complication.) But, if what one is interested in is the calculation of possible values of the observable, the interpretation of the theorem may be quite irrelevant.*

An inconsistency of such a formal system may be embarrassing; for instance, it could be proved (in one way) that λ is and (in another way) that λ is not an eigenvalue. Even that need not necessarily make the formal system useless: we might order proofs according to their lengths, so that if there are proofs of Π and of $\neg \Pi$, only the shorter one counts. This would give a classification of formulae, which is what is needed in the applications mentioned above, and in some application it may be the right one.

G. K.

* The crux of the matter is: the experimental proposition " λ is a value of the observable" is associated with the formulae $\mathcal{O}(\lambda)$, where $\mathcal{O}(n)$ is a bound variable formula.

1662. One scene comes back to me, a clear vision out of the mists of the forgotten past. A little boy of six, seated at a table in the dining-room of the Chief Secretary's Lodge, crying over a long division sum (I remember it was long division) and an angel in human shape coming to comfort me and show me how to work out the abstruse problem. It was dear Dolly Tennant, afterwards wife of Stanley the great explorer.—G. M. Trevelyan, *An Autobiography*, p. 7.

SMALL OSCILLATIONS WITH DAMPING

BY JOHN WILLIAMS.

Introduction.

Most textbooks in dynamics deal with the problem of simple harmonic motion when there is damping present proportional to the velocity; the solution in this case is exact. If, on the other hand, we prescribe damping proportional to the square of the velocity the solutions, in general, can not be evaluated in finite terms, and so we have to resort to approximations. Since approximations play such a leading role in many current problems in applied mathematics, I feel that familiarity with approximations is an essential part of the early training of a good applied mathematician, whether he wishes to be a theorist or a practical engineer. The subject should be broached early. For instance, the following analysis should not be beyond a scholarship student and might be used to supplement his textbook, and at the same time help him to get rid of the artificial idea that all useful solutions are necessarily exact.

The equations of motion and the first method of solution.

Let x measure the displacement of a particle from a fixed point O along a line Ox , the particle moving under a restoring force n^2x and a resistance to motion $\frac{1}{2}\mu\dot{x}^2$, where μ is a constant.* If we take the mass as unity, since this involves no loss of generality, the equations of motion are

$$\ddot{x} - \frac{1}{2}\mu\dot{x}^2 + n^2x = 0, \quad \text{when } \dot{x} \leq 0, \dots\dots\dots(1.1)$$

$$\ddot{x} + \frac{1}{2}\mu\dot{x}^2 + n^2x = 0, \quad \text{when } \dot{x} \geq 0. \dots\dots\dots(1.2)$$

These equations may, of course, be combined into the single equation

$$\ddot{x} + \frac{1}{2}\mu\dot{x} |\dot{x}| + n^2x = 0. \dots\dots\dots(2)$$

If we assume that at $t=0$, $\dot{x}=0$ and $x=a$ (>0), then, for the initial stages of the motion $\dot{x}<0$, we use (1.1) until \dot{x} is zero again. Putting $\dot{x}=v$, this equation becomes

$$v \frac{dv}{dx} - \frac{1}{2}\mu v^2 + n^2x = 0.$$

This can be integrated in finite terms as

$$v^2 = 2n^2 e^{\mu x} \int_x^a x e^{-\mu x} dx,$$

$$\text{or} \quad \mu^2 v^2 = 2n^2 \{(\mu x + 1) - (\mu a + 1) e^{-\mu(a-x)}\}. \dots\dots\dots(3)$$

The first stationary position.

One value of x for which $v=0$ is obviously $x=a$. To find the other we have to approximate. When $\mu=0$, the required value of x is $-a$. Putting $a-x=y$, we have

$$v^2 = n^2 y (2a - (\mu a + 1)y + \frac{1}{2}\mu(\mu a + 1)y^2 - \frac{1}{6}\mu^2(\mu a + 1)y^3 + \dots) \dots\dots\dots(4)$$

When μa is small (and it should be observed that its dimensions are zero, so that it is a pure number), it is evident that the required root of $v^2=0$ gives y in the form

$$y = 2a \{1 + \alpha_1 \mu a + \alpha_2 \mu^2 a^2 + O(\mu^3 a^3)\}, \dots\dots\dots(5)$$

where α_1 and α_2 are constants to be determined. Using this value of y in (4)

* It is worth while observing that the dimensions of μ are those of inverse length namely, L^{-1} .

with $v=0$, and having agreed to neglect quantities of order $\mu^3 a^3$, since μa is small in practice, we have, avoiding the solution $y=0$,

$$0 = 2a - (\mu a + 1) 2a (1 + \alpha_1 \mu a + \alpha_2 \mu^2 a^2) + \frac{1}{2} \mu (\mu a + 1) 4a^2 (1 + 2\alpha_1 \mu a) - \frac{1}{2} \mu^3 (\mu a + 1) 8a^3.$$

Whereupon, equating the coefficients of μa and $\mu^2 a^2$ to zero, we have

$$\alpha_1 = -\frac{1}{2}, \quad \alpha_2 = \frac{2}{9}.$$

Hence if $\mu^3 a^3$ is small enough to be neglected in comparison with unity, the first stationary position is given by

$$a - x = y = 2a (1 - \frac{1}{2} \mu a + \frac{2}{9} \mu^2 a^2). \dots\dots\dots (6)$$

The time to reach the first stationary position.

To find the time which the particle takes to reach this latter stationary position we have, to the same degree of approximation,

$$-\frac{dx}{dt} = v = \frac{dy}{dt} = n \sqrt{y(2a - (\mu a + 1)y + \frac{1}{2} \mu (\mu a + 1)y^2 - \frac{1}{2} \mu^3 (\mu a + 1)y^3)}. \dots (7)$$

Now, since to the degree of approximation stated the value of y given in (6) is a root of the expression under the square root sign in (7), this expression can be put in the form

$$\{2a(1 - \frac{1}{2} \mu a + \frac{2}{9} \mu^2 a^2) - y\} (\beta_0 + \beta_1 y + \beta_2 y^2).$$

Again, neglecting quantities of the order of $\mu^3 a^3$ and equating coefficients of other powers of μa , we have

$$\beta_2 = \frac{1}{2} \mu^2, \quad \beta_1 = -(\frac{1}{2} \mu + \frac{1}{9} \mu^2 a), \quad \beta_0 = 1 + \frac{1}{2} \mu a - \frac{1}{9} \mu^2 a^2.$$

To the same degree of approximation we readily obtain

$$\begin{aligned} (\beta_0 + \beta_1 y + \beta_2 y^2)^{-\frac{1}{2}} &= (1 + \frac{1}{2} \mu (a - y) - \frac{1}{2} \mu^2 (4a^2 + 6ay - 3y^2))^{-\frac{1}{2}} \\ &= 1 - \frac{1}{4} \mu (a - y) + \frac{7}{72} \mu^2 a^2. \end{aligned}$$

Hence, writing (6) in the form $y = 2a(1 - \epsilon)$, the required time is τ where

$$n\tau = \int_0^{2a(1-\epsilon)} [1 - \frac{1}{4} \mu (a - y) + \frac{7}{72} \mu^2 a^2] \cdot [y(2a(1 - \epsilon) - y)]^{-\frac{1}{2}} \cdot dy.$$

Using the substitution $y = 2a(1 - \epsilon) \sin^2 \theta$ this integral becomes

$$n\tau = 2 \int_0^{\pi/2} \{1 + \frac{7}{72} \mu^2 a^2 - \frac{1}{6} \mu a (1 - 2 \sin^2 \theta - \frac{1}{2} \mu \epsilon a \sin^2 \theta)\} d\theta.$$

But

$$\int_0^{\pi/2} (1 - 2 \sin^2 \theta) d\theta = 0, \quad \int_0^{\pi/2} \sin^2 \theta d\theta = \frac{1}{4} \pi,$$

and $\mu \epsilon a = \frac{1}{2} \mu^2 a^3 + O(\mu^3 a^3)$ so that, to the required degree of approximation,

$$\tau = \frac{\pi}{n} \left(1 + \frac{a^2 \mu^2}{24}\right). \dots\dots\dots (8)$$

In the motion from the first stationary position given by (6) we have to use the equation of motion as given by (1.2), for now \dot{x} will be positive until it becomes zero again. Corresponding to (6), which gives

$$x = -a(1 - \frac{2}{3} \mu a + \frac{4}{9} \mu^2 a^2) = -a', \text{ say,}$$

we have the second stationary position given by

$$\begin{aligned} x &= a'(1 - \frac{2}{3} \mu a' + \frac{4}{9} \mu^2 a'^2) \\ &= a(1 - \frac{2}{3} \mu a + \frac{4}{9} \mu^2 a^2) + O(\mu^3 a^3). \dots\dots\dots (9) \end{aligned}$$

The time will, to the same degree, remain the same, so that the total time for a complete cycle, that is, the period, is

$$\frac{2\pi}{n} \left(1 + \frac{a^2 \mu^2}{24} \right).$$

The second method of solution.

It is interesting, besides being a useful check on our work, to obtain the solution for x in terms of t over the region $\dot{x} \leq 0$, under the same conditions, namely, that we neglect quantities of order $\mu^2 a^3$. When $\mu = 0$, the solution of (1.1) subject to the initial conditions $\dot{x} = 0$, $x = a$, when $t = 0$, is obviously $x = a \cos nt$. When $\mu \neq 0$, but $\mu^2 a^3$ can be neglected, we assume tentatively a solution of the form

$$x = a (\cos nt + \mu a x_1 + \mu^2 a^2 x_2), \dots \dots \dots (10)$$

where x_1 and x_2 are functions of time to be determined and are such that $x_1 = x_2 = \dot{x}_1 = \dot{x}_2 = 0$ when $t = 0$. Admitting this form of solution into equation (1.1) we find, equating powers of μa , that x_1 and x_2 satisfy the equations

$$\ddot{x}_1 + n^2 x_1 = \frac{1}{2} n^2 \sin^2 nt, \dots \dots \dots (11)$$

$$\ddot{x}_2 + n^2 x_2 = -n \dot{x}_1 \sin nt. \dots \dots \dots (12)$$

Using Laplace transform methods, or otherwise, it is quite straightforward to deduce the solution of (11) subject to the initial conditions as

$$x_1 = \frac{1}{12} (3 - 4 \cos nt + \cos 2nt). \dots \dots \dots (13)$$

Equation (12) now assumes the form

$$\begin{aligned} \ddot{x}_2 + n^2 x_2 &= -\frac{1}{12} n \sin nt (4n \sin nt - 2n \sin 2nt) \\ &= -\frac{1}{12} n^2 (2 - \cos nt - 2 \cos 2nt + \cos 3nt). \end{aligned}$$

The solution of this equation subject to the initial conditions is

$$x_2 = (-48 + 61 \cos nt + 12n \sin nt - 16 \cos 2nt + 3 \cos 3nt)/288. \dots \dots \dots (14)$$

It should be noticed that n may be as large as it pleases since we are only concerned with the range $0 \leq nt \leq n\tau$, as given in (8).

The complete expression for x is, of course, found by using (10) in conjunction with (13) and (14). On the other hand, we find that

$$\begin{aligned} \dot{x} &= -na \sin nt + \frac{1}{6} na^2 \mu (2 \sin nt - \sin 2nt) \\ &\quad + na^3 \mu^2 (-61 \sin nt + 12 \sin nt + 12 nt \cos nt + 32 \sin 2nt \\ &\quad - 9 \sin 3nt)/288. \end{aligned}$$

The roots of $\dot{x} = 0$ are either $t = 0$ or $nt = n\tau = \pi + \gamma_1 a \mu + \gamma_2 a^2 \mu^2$, say. Hence, neglecting $\mu^3 a^3$,

$$0 \equiv \gamma_1 a \mu + \gamma_2 a^2 \mu^2 + \frac{1}{6} \mu a (-4 \gamma_1 a \mu) - \frac{1}{24} \pi a^3 \mu^3,$$

and equating coefficients, $\gamma_1 = 0$, $\gamma_2 = \frac{1}{24} \pi$.

These results agree with the value of τ given in (8). Using this value of τ we have

$$\cos n\tau = -1 + O(a^3 \mu^3),$$

$$x_1 = \frac{2}{3} + O(a^3 \mu^3),$$

$$x_2 = -\frac{4}{9} + O(a \mu).$$

Hence at this value of τ , using (10),

$$x = a (-1 + \frac{2}{3} \mu a - \frac{4}{9} \mu^2 a^2),$$

a result which agrees with (6).

The interested reader may care to investigate for himself the problem of finding when and where \dot{x} is a maximum.

J. W.

NON-LINEAR DIFFERENTIAL EQUATION HAVING A PERIODIC COEFFICIENT.

BY N. W. MCLACHLAN.

1. *Introduction.*

Consider a thin thread (preferably nylon, which is strong, has low loss, and is resilient) attached to an electromagnetically-driven reed or tuning fork at one end, the other passing over a pulley, and tensioned by a weight W . Suppose the free length of the thread is about 1 metre, and that the reed vibrates *transversely* to the thread. By adjusting W to get the correct tension, the first overtone of the thread may be obtained, there being a node* at the middle. The frequency of the thread is the same as that of the reed. If the latter is rotated through 90° (the length of the thread being unaltered), vibration occurs *along* the thread, thereby causing its tension to vary periodically. The main frequency of the thread is now *half* that of the reed, so a subharmonic of order two occurs, there being no node at the middle. This was demonstrated by F. Melde ninety years ago,† using a tuning fork excited by a violin bow.

A more versatile version of the experiment is achieved by driving the reed from the 50 c.p.s. electric supply mains,‡ using a variac or a rheostat to control the current, and therefore the amplitude X of the reed. Starting with X very small, the thread does not vibrate, but if X is increased gradually, a threshold is reached when the thread begins to show signs of life. If now X is increased suddenly, the amplitude of the thread will grow rapidly until it attains a constant value, the motion then being periodic, with period $1/50$ th second. By putting one's ear in a *suitable position*, well away from the reed (the sound from which may be screened), the subharmonic is the only tone *audible*, so the motion is sensibly sinusoidal. It is interesting to view the thread using intermittent illumination, e.g. by aid of a stroboscope. The "visual motion" may then be slowed down to any desired extent. If the frequency variation of both reed and stroboscope is small enough, the configuration at any instant during a period may be "frozen", i.e. it appears to be stationary.

In what follows we shall investigate the above observations analytically: (i) the amplitude growth, (ii) the ultimate periodic motion, (iii) the threshold.

2. *The differential equation.*

To effect simplification we commence with the case where inherent loss is neglected. Then at any point between the ends of the thread, the equation for lateral displacement y is §

$$d^2y/dt^2 + (\pi^2 T_0/ml^2)(1 - 2\gamma \cos 2\omega t)y = 0, \dots\dots\dots(1)$$

where T_0 is the statical tension, m the mass per unit length, l the length, γ a constant < 0.5 , and $\omega = 2\pi \times 50$ radians per second. During vibration the tension is given by $T_0(1 - 2\gamma \cos 2\omega t)$, and it varies over the range $T_0(1 \pm 2\gamma)$. If $\gamma = 0$, (1) has the well-known solution

$$y = A \sin \nu t + B \cos \nu t, \dots\dots\dots(2)$$

* A true node cannot occur in a dissipative system owing to energy loss. It is a position of least amplitude.

† Poggendorff's *Annalen*, 109, 193, 1860.

‡ The reed frequency is 100 c.p.s., since the magnetic attraction is independent of the direction of the current through the magnet winding.

§ McLachlan, *Theory and Application of Mathieu Functions*, p. 289, which will be designated by M.F.

where A, B are arbitrary constants, $\nu = \pi(T_0/ml^2)^{1/2}$, and the periodic time is $2l(m/T_0)^{1/2}$.

Writing $z = \omega t$, $a = \pi^2 T_0 / ml^2 \omega^2$, $q = \gamma \pi^2 T_0 / ml^2 \omega^2 = \gamma a$, (1) becomes

$$d^2y/dz^2 + (a - 2q \cos 2z)y = 0, \dots\dots\dots(3)$$

which is the canonical form of Mathieu's equation. In our case $\pi^2 T_0 / ml^2 = \omega^2$, giving $a = 1$, so we have to discuss the solution of

$$y'' + (1 - 2q \cos 2z)y = 0, \dots\dots\dots(4)$$

$$0 < q \leq 1.$$

3. Solution of (4), § 2.

Referring to the stability chart M.F., Fig. II, p. 98, the parametric point $(1, q)$ lies in an unstable region of the (a, q) plane. Thus by M.F. (3), 2°, § 4.70, one solution takes the form

$$y_1(z) = e^{i\mu z} \sum_{-\infty}^{\infty} c_{2r+1} e^{(2r+1)iz}, \quad \mu > 0, \dots\dots\dots(1)$$

$$= e^{i\mu z} \times \text{bounded periodic function.} \dots\dots\dots(2)$$

Hence $y_1 \rightarrow \pm \infty$ as $z \rightarrow +\infty$, and the amplitude of vibration increases without limit as $t \rightarrow +\infty$. In practice, however, the amplitude attains an ultimate value, so (4), § 2 accounts for the growth, but not for the limitation.

4. Modification of (3), § 2.

We may consider a driven thread to be a system in which the restoring force is related linearly to displacement, so long as the latter is small. But in the Melde experiment the amplitude may be several centimetres, so the possibility of non-linear control arises. Referring to (2), § 5, *Mathematical Gazette*, 32, 64 (1948), in a system having control ay , $y \rightarrow \pm \infty$ as $\omega^2 \rightarrow a$, at the resonance or tune point. But in a non-linear system with control $ay + by^3$, by (13), § 3, *loc. cit.*, there is no finite ω for which resonance occurs. Now the displacement of the thread is symmetrical about its central position and, therefore, if the control is $f(y)$, we must have $f(y) = -f(-y)$, an odd function of y . Accordingly for present purposes we shall take $f(y) = ay + by^3$, $a, b > 0$, and determine the effect of the non-linear term by^3 . Then instead of (3), § 2, we get the non-linear equation

$$y'' + \{(a + by^2) - 2q \cos 2z\}y = 0, \dots\dots\dots(1)$$

and seek a periodic solution having period 2π . This may be found either by assuming $y = A_1 \cos z + A_3 \cos 3z + \dots$, or by the method of perturbation as exemplified in M.F., § 2.13 *et seq.* Choosing the latter, with b, q small and $q = b\epsilon$, we assume that

$$y = y_0 + by_1 + b^2y_2 + \dots, \dots\dots\dots(2)$$

and

$$a = 1 + \alpha_1 b\epsilon + \alpha_2 b^2\epsilon^2 + \dots, \dots\dots\dots(3)$$

the y_n being twice differentiable periodic functions of z , and the α_n constants. The initial conditions $y(0) = A$, $y'(0) = 0$, entail $y_n(0) = y'_n(0) = 0$, $n = 1, 2, \dots$, since the solution is to hold in $0 < b \leq b_0$.

Then to order two in b , we find that

$$\begin{aligned} y = & A\{1 + (q/8)(1 - A^2/4\epsilon) + (5q^2/192)(1 + 3A^2/20\epsilon)\} \cos z \\ & - (Aq/8)(1 - A^2/4\epsilon)(1 + q/4) \cos 3z \\ & + (Aq^2/192)(1 - A^2/4\epsilon)(1 - 3A^2/4\epsilon) \cos 5z, \dots\dots\dots(4) \end{aligned}$$

and

$$a = (1 - 3A^2b/4 + 3A^4b^2/128) + q(1 - A^2b/16) - q^2/8. \dots\dots\dots(5)$$

(4) contains no term in $\cos 2z$, so the motion has no component with the reed period. Now $a = 1$, so by (5), in the first approximation,

$$|A| = (4q/3b)^{\frac{1}{2}}. \dots\dots\dots(6)$$

(5) yields another value of $|A|$, but it is inadmissible since the first initial condition cannot be reproduced. Inserting (6) into (4), and taking A positive, we obtain

$$y \simeq A \{ (1 + q/12 + q^2/32) \cos z - (1/12)(q + q^2/4) \cos 3z \}, \dots\dots\dots(7)$$

which gives

$$y(0) \simeq A(1 - q^2/96) \simeq A, \dots\dots\dots(8)$$

as required, since q is small.

From (7) it follows that the amplitude ratio of the third harmonic of the thread motion to its fundamental is approximately $(q/12)(1 + q/6)$. When $q = 0.25$, the ratio is nearly 0.022. Hence the motion is almost sinusoidal, which agrees with experimental observation. This seems to be a feature of certain dynamical systems having non-linear control.

At the commencement of the motion, when the amplitude is small, by^2 is negligible and the control is due mainly to ay . Moreover, the unstable Mathieu solution (1), § 3, holds approximately in the initial stage. As the amplitude grows, so also does the influence of by^2 , until ultimately it causes limitation and stability, in virtue of distuning. By (6), $|A|$ decreases with increase in b , and *vice versa*.

In solving (1) if we keep the coefficient of $\cos z$ the same throughout—which entails an initial condition slightly different from that above, since the D.E. is non-linear—then

$$y = A \{ \cos z - (q/8)(1 - A^2/4\epsilon) [1 + (q/8)(1 + 3A^2/4\epsilon) \cos 3z + (q^2/192)(1 - A^2/4\epsilon)(1 - 3A^2/4\epsilon) \cos 5z], \dots\dots\dots(9)$$

and

$$a = 1 + q(1 - 3A^2/4\epsilon) - (q^2/8)(1 - A^2/4\epsilon)(1 - 3A^2/4\epsilon). \dots\dots\dots(10)$$

When $A = 1$ and $b = 0$, (9) degenerates to the early terms in the series for the Mathieu function $ce_1(z, q)$, and (10) degenerates to those in its characteristic number a_1 (M.F., p. 13). If $a = 1$, (10) yields $3A^2/4\epsilon = 1$ or $|A| = (4q/3b)^{\frac{1}{2}}$, as before.

Writing $A^2 = 4\epsilon = 4q/b$ in (9), the coefficients of $\cos 3z$, $\cos 5z$ vanish, and we obtain the *exact* subharmonic solutions

$$y = \pm 2(q/b)^{\frac{1}{2}} \cos z,$$

while by (10) the condition to be satisfied is that $a = (1 - 2q)$, so $a < 1$.

5. Influence of energy dissipation.

To include this we add the damping term $2\kappa y'$ to (1), § 4, κ being small > 0 , and obtain

$$y'' + 2\kappa y' + \{ (a + by^2) - 2q \cos 2z \} y = 0. \dots\dots\dots(1)$$

Since the motion is known to be almost sinusoidal, as a first approximation we take

$$y = A_1 \cos z + B_1 \sin z, \dots\dots\dots(2)$$

the sine term being needed to allow for a phase change caused by loss. Substituting into (1) and equating the coefficients of $\cos z$, $\sin z$ to zero independently, with $A = (A_1^2 + B_1^2)^{\frac{1}{2}}$, this being the amplitude, we get

$$A_1(a - 1 - q + 3bA^2/4) + 2\kappa B_1 = 0, \dots\dots\dots(3)$$

and

$$-2\kappa A_1 + (a - 1 + q + 3bA^2/4)B_1 = 0. \dots\dots\dots(4)$$

Equating the values of A_1/B_1 from (3), (4) leads to

$$(a - 1 + 3bA^2/4)^2 = (q^2 - 4\kappa^2), \quad \dots\dots\dots(5)$$

and since $a = 1$, we obtain

$$A^2 = (4/3b)(q^2 - 4\kappa^2)^{1/2}, \quad \dots\dots\dots(6)$$

so

$$|A| = (4/3b)^{1/2}(q^2 - 4\kappa^2)^{1/4}. \quad \dots\dots\dots(7)$$

For the reality of A we must have $2\kappa < q$, which is the condition for maintenance of the vibration. The threshold is $q = 2\kappa$, and if $0 < q < 2\kappa$, there is no motion. If, however, the thread were set in vibration by hand, the motion would decay and be extinguished ultimately. Thus the existence of a threshold is due to loss, as one would surmise on physical grounds.

In the first approximation, by (2) the displacement of the thread may be written

$$y = A \cos(\omega t - \theta_1), \quad \dots\dots\dots(8)$$

where $\theta_1 = \tan^{-1} B_1/A_1 = \tan^{-1} 2\kappa/(q + \sqrt{q^2 - 4\kappa^2}) \simeq \kappa/q$, if $\kappa \ll q$. The second approximation yields a term in $\cos(3\omega t - \theta_3)$, whose amplitude is of an order similar to that of $\cos 3z$ in (7), §4.

6. Variation in length of thread.

During normal vibration, the length of the thread may be varied by moving the pulley towards or away from the reed. This corresponds to an alteration in $a = \pi^2 T_0 / ml^2 \omega^2$. For the vibration to continue, A must be real, so by (5), §5, with $q = \gamma a$, the inequality

$$\gamma > \{(a - 1)^2 + 4\kappa^2\}^{1/2} / a, \quad \dots\dots\dots(1)$$

must be satisfied. If an equality sign is used, we obtain the limiting values of a when the motion ceases. Solving for a , we get

$$a = \{1/(1 - \gamma^2)\} \{1 \pm [\gamma^2(1 + 4\kappa^2) - 4\kappa^2]^{1/2}\}, \quad \dots\dots\dots(2)$$

so with $0.5 > \gamma \gg 2\kappa$,

$$a \simeq 1/(1 - \gamma), \quad \text{and} \quad 1/(1 + \gamma). \quad \dots\dots\dots(3)$$

Taking $\gamma = 1/4$, the limits are $4/3$ and $4/5$, so if l_1 is the length of thread corresponding to $a = 1$, l may be varied from $\frac{1}{2}l_1 \sqrt{3}$ to $\frac{1}{2}l_1 \sqrt{5}$, a total range of about $l_1/4$.

7. Additional solutions of (1), §4.

If the statical tension of the thread is increased fourfold, we write 4 for 1 in (3), §4. Keeping the coefficient of $\cos 2z$ the same throughout, the solution to order two in b , is

$$y = A \{ (q/4)(1 - A^2b/8) + \cos 2z - (q/12)(1 - 13A^2b/512) \cos 4z \\ + (1/384)[q^2 + 3A^2b + (9/128)A^4b^2] \cos 6z \}, \quad \dots\dots\dots(1)$$

$$\text{and} \quad a = 4 - 3A^2b/4 + 5q^2/12 - 3A^4b^2/512. \quad \dots\dots\dots(2)$$

When $A = 1$ and $b = 0$, these formulae degenerate to the early terms in $ce_4(z, q)$ and a_2 (M.F., p. 15). With $a = 4$, $A \simeq (q/3)(5/b)^{1/2} < (4q/3b)^{1/2}$, $0 < q \leq 1$, as would be expected in virtue of the greater tension in the present case. The solution (1) contains a unidirectional term $A\{(q/4)(1 - A^2b/8) \simeq Aq/4$, so the "centre of oscillation" is displaced from the rest position to this extent. No subharmonic occurs, and the dominant component has the same period as the reed. The amplitude ratio of the first overtone to the fundamental is nearly $q/12 \simeq 0.02$ if $q = 0.25$, which is practically the same as that in (9), §4.

For a ninefold increase in tension we write 9 for 1 in (3), §4, and proceed as hitherto. Then, to order two in b , we find that

$$y = A \{ \cos 3z + (q/8)[1 + (q/8)(1 - 3A^2b/8q)] \cos z - (q/16)(1 - 3A^2b/64) \cos 5z \}, \dots\dots\dots(3)$$

and

$$a = 9 - 3A^2b/4 + (q^2/16)(1 - A^2b^2/24). \dots\dots\dots(4)$$

When $A = 1$ and $b = 0$, these degenerate to the early terms in $ce_3(z, q)$ and a_3 . If $a = 9$, then $A \simeq q/2(3b)^{1/2} < (q/3)(5b)^{1/2} < (4q/3b)^{1/2}$, $0 < q \leq 1$, so with q assigned the amplitude of the dominant term decreases with increase in tension—remembering that we are dealing with discrete increments. Absence of a term in $\cos 2z$ signifies that the motion has no component of reed period. There is, however, a subharmonic, but when $q = 0.25$, its amplitude ratio to that of the term $\cos 3z$ is only 0.031. This contrasts strongly with the ratio $1/0.022 \simeq 45.4$ in § 4 with $a = 1$.

From above it follows that if $b > 0$, (1) § 4 has solutions of positive integral order 1, 2, Those of odd order contain a subharmonic having period 2π , but there is no component having the same period as the reed. The lower the order, the greater is the relative amplitude of the subharmonic. The solutions of even order have no subharmonic, but they contain a unidirectional term and also a component whose period is the same as that of the reed. The lower the order, the greater is the relative amplitude of this component.

N. W. McL.

1663. How proportional magnitudes ought to be defined, is still a subject of controversy among geometers. Euclid defined them thus: "The first of four magnitudes, etc., . . ." This definition is liable to the objection that there is not the least resemblance between it and the common notions of similitude or equality of ratios; it must be confessed indeed, that Euclid has demonstrated that magnitudes thus related retain the same properties, however they may be inverted or subjected to the various changes made use of by geometers; but this is not sufficient: the connexion ought to be shewn between this definition and that relation of two ratios to which the name of equality is commonly given. Against this objection Barrow ably argues in his mathematical lectures, and after having overturned the various theories invented since the time of Euclid, endeavoured to put a stop to the controversy, and repress every attempt towards forming a new theory, in the following words: "Per has naturas [scil. magnitudinum proportionalium] nil aliud intelligi quam rei definitae nomini, quatenus in usu communi versatur, respondentis conceptus aut significatus aliquos imperfectos et indistinctos, in scientiis minime respiciendos, ad quos proinde nullatenus exigendae sunt definitiones; imo succedendis et eliminandis iis, ipsorumque loco substituendis rerum certis distinctis atque claris ideis, efformantur definitiones." If this be true, Euclid's definition is certainly the best, for undoubtedly it is wholly foreign from common use. . . .—*The First Six Books of the Elements of Euclid*, by Thomas Elrington, Trin. Coll. Dub. 12th edition 1844 (first English edition published in 1801). [Elrington uses as his definition of proportion the following: Magnitudes are said to be in the same ratio, the first to the second as the third to the fourth, when any submultiple whatsoever of the first is contained in the second, as often as an equi-submultiple of the third is contained in the fourth. In consequence his version of Bk. V contains 39 theorems.] [Per Mr. B. A. Swinden.]

APPROXIMATE METHODS IN ELEMENTARY MATHEMATICS.

By J. TOPPING.

METHODS of approximation, and successive approximation, are often used in more advanced mathematics, but it is not widely realised that they can with advantage be introduced at quite an early stage in the teaching of school mathematics. This note is a plea that these methods should be taught earlier; indeed it is written in the belief that a wider use of these methods would have not only a salutary effect on teaching and learning alike, but would help to eliminate some of the widespread troubles with which teachers of science are generally confronted and with which they often ask us to help. How often are we told by science colleagues that pupils calculate a physical quantity, for instance, a specific heat, to four decimal places when it is likely to be in error in the second place! Similarly those of us who teach practical mechanics, know full well how the resources of four- or five-figure logarithmic tables are invariably used to the full, when the data are correct only to two significant figures.

In the teaching of mathematics generally students are too often presented with precise data to which a particular mathematical technique is expected to be applied, and no question arises as to the degree of accuracy of the result; it is either right or wrong. It is only wrong if there is an error in the technique. In the world of experiment, however, precise data are unknown; errors of observation and the like are always present, so that calculations involving measured data lead to results which are inevitably inaccurate, even though the technique of calculation has been correctly applied. The accuracy of the result is dictated by the accuracy of the data, and consequently the complete process of calculation is often neither justifiable nor necessary. An approximate solution is usually adequate. In addition, there are problems, whatever the accuracy of the data, to which a complete mathematical solution cannot be found and an approximate solution the only possible one.

An essential feature of approximate methods is the assessment of the relative values of groups of quantities involved; familiarity with this technique should help, it is hoped, to inculcate a balanced attitude towards numerical data generally.

Square Roots.

An easy introduction to approximate methods may be made by evaluating the square root of a simple positive number, N . It is a method well known to users of calculating machines.

As N can be expressed as the product of two positive factors a and b , it is clear that the square root of N must lie between a and b ; as a first approximation it is natural to take the arithmetic mean of a and b .

We can write $\sqrt{N} = \sqrt{(ab)} \approx \frac{1}{2}(a+b)$. This is obvious to those who have done even very little algebra; indeed it is clear that the nearer to equality a and b are the better the approximation.

Of course, we can establish this by writing

$$\sqrt{ab} = \frac{1}{2}\sqrt{\{(a+b)^2 - (a-b)^2\}},$$

and, if appropriate, link this with the usual work on arithmetic and geometric means. But long before these topics are normally discussed the approximation $\frac{1}{2}(a+b)$ can be used to evaluate the square roots which arise.

For example, $3 = 2 \times 1.5$. Hence $\sqrt{3} \approx \frac{1}{2}(2 + 1.5) = 1.75$.

Moreover, the process can be repeated, for

$$3 = 1.75 \times \frac{12}{7} = 1.75 \times 1.714.$$

Thus

$$\sqrt{3} \simeq \frac{1}{2}(1.75 + 1.714) = 1.732.$$

Another example of wide application is $\sqrt{1+x}$, where x is small compared with unity.

We have

$$\begin{aligned}\sqrt{1+x} &= \sqrt{1 \times (1+x)} \\ &\simeq \frac{1}{2}(1 + 1+x) = 1 + \frac{1}{2}x.\end{aligned}$$

How unfortunate it is that we so often wait until the Binomial Theorem has been tackled before introducing this approximation.

Similarly,

$$\begin{aligned}\sqrt{4+x} &= \sqrt{2(2 + \frac{1}{2}x)} \quad \text{or} \quad 2\sqrt{1 + \frac{1}{4}x} \\ &\simeq 2 + \frac{1}{4}x \quad \text{if } x \ll 4.\end{aligned}$$

This approximation method may also help to avoid errors in calculating square roots of numbers less than unity. For example,

$$\sqrt{0.95} = \sqrt{1 \times 0.95} \simeq 0.975$$

and

$$\sqrt{0.095} = \sqrt{0.3 \times 0.316} \simeq 0.308.$$

But perhaps enough examples have been given to illustrate the power and simplicity of the method.

Solution of Equations.

The evaluation of \sqrt{ab} gives, of course, the roots of the equation $x^2 = ab$.

A method of approximation can be applied, however, to solve any quadratic equation. For example, consider the equation

$$x^2 - 10x + 1 = 0. \dots\dots\dots(1)$$

We can neglect any term or terms in this (or any other) equation provided we can justify such a step; we might proceed as follows, the justification being *à posteriori*.

Suppose we neglect the x^2 term, so that the equation becomes

$$-10x + 1 = 0;$$

that is,

$$x = \frac{1}{10}.$$

Can this be justified? Substituting the value $x = \frac{1}{10}$ in the original equation (1) the terms on the left-hand side are $\frac{1}{100}$, -1 , $+1$ respectively, so that in neglecting the first term we have effectively neglected $\frac{1}{100}$ when the other terms are each numerically unity. We may therefore expect the approximation $x = \frac{1}{10}$ to be accurate to about 1 per cent.

Our first approximation is therefore $x = 0.1$. But we can obtain closer approximations. The equation can be written

$$x = \frac{1}{10} + \frac{1}{10}x^2,$$

and in the second term on the right-hand side we can obviously substitute $x \simeq 0.1$. Thus as a second approximation we get

$$x \simeq 0.1 + 0.001 = 0.101.$$

But the reader may ask, as the brighter student certainly will, "why neglect x^2 ?; why not neglect either or both of the other terms in the equation?" Let us try.

Neglecting $+1$ the equation becomes

$$x^2 - 10x = 0.$$

Thus

$$x = 0 \quad \text{or} \quad 10.$$

Now if $x = 0$ the terms in the equation (1) are 0, 0, 1, but 1 cannot be

neglected compared with 0, and hence $x=0$ must be rejected. However, if $x=10$ the terms in the equation (1) are 100, -100, 1, which suggest that the approximation $x=10$ is accurate to about 1 per cent.

Again, we can find a closer approximation by writing

$$x^2 = 10x - 1.$$

Hence

$$x = 10 - \frac{1}{x}.$$

Hence substituting $x=10$ in the right-hand side we get as a second approximation :

$$x \approx 10 - \frac{1}{10} = 9.9.$$

The process can be repeated if higher accuracy is required ; *e.g.*

$$x \approx 10 - \frac{1}{9.9} = 9.899.$$

Similarly, if the term $-10x$ is neglected equation (1) becomes $x^2 + 1 = 0$, which has no real solution. No other solutions are possible.

The roots of the quadratic equation are therefore 0.10 and 9.90 approximately.

This method of deriving approximately the roots of an equation appeals, even if initially it is somewhat startling, to students already familiar with the formal method of solving a quadratic. The writer has not had the opportunity of trying it with younger pupils ; it would be interesting to do so.

One additional advantage, from the point of view of school mathematics, is that it can be applied to an equation of higher degree, *e.g.* a cubic equation, such as

$$x^3 - 10x + 1 = 0. \dots\dots\dots(2)$$

Neglecting x^3 we get $x = \frac{1}{10}$, and on substitution we find it is accurate to about 0.1 per cent. A second approximation is given by

$$\begin{aligned} x &= \frac{1}{10} + \frac{1}{10}x^2 \\ &\approx \frac{1}{10} + \frac{1}{10}\left(\frac{1}{10}\right)^2 = 0.1001. \end{aligned}$$

Thus one root is obtained to four decimal places very quickly.

Similarly, if we neglect the term $+1$ we get

$$x^3 - 10x = 0.$$

Thus

$$x = 0, \quad \sqrt{10}, \quad -\sqrt{10}.$$

On substituting these values in the left-hand side of equation (2) it is found that the value $x=0$ must be rejected, and the values $x = \pm\sqrt{10}$ are accurate to about 1 in $10\sqrt{10}$, *i.e.* to about 3 per cent.

Proceeding, equation (2) can be written

$$x^2 - 10 = -1/x,$$

and therefore

$$x = \pm \sqrt{\left(10 - \frac{1}{x}\right)}.$$

Hence a closer approximation to the root near $x = \sqrt{10}$ is

$$x \approx +\sqrt{\left(10 - \frac{1}{\sqrt{10}}\right)} = +\sqrt{9.684} = 3.11,$$

and a closer approximation to the root near $x = -\sqrt{10}$ is

$$x \approx -\sqrt{\left(10 + \frac{1}{\sqrt{10}}\right)} = -\sqrt{10.316} = -3.21.$$

The process can obviously be repeated if greater accuracy is required. We have found the three roots 0.10, 3.11 and -3.21 approximately.

It is instructive to compare this with any other method of solution of a cubic equation. It will be noted, of course, that equation (2) has been so chosen as to demonstrate the more effectively the power of this method of successive approximation, but it can be applied, often very economically, to other equations, including equations of higher degree.

It is perhaps worth remarking that (2) might be tackled in another way.

Since $x^3 - 10x + 1$ is negative, (-2) , when $x = 3$ and positive, $(+25)$, when $x = 4$, there is a root of the equation between $x = 3$ and $x = 4$, and near to $x = 3$.

Writing $x = 3 + y$ we get $(3 + y)^3 - 10(3 + y) + 1 = 0$.

Hence

$$y^3 + 9y^2 + 17y - 2 = 0,$$

and so

$$\begin{aligned} y &= \frac{2}{17} - \frac{9}{17}y^2 - \frac{1}{17}y^3 \\ &\simeq \frac{2}{17} - \frac{9}{17}\left(\frac{2}{17}\right)^2 - \frac{1}{17}\left(\frac{2}{17}\right)^3 \\ &\simeq 0.118 - 0.007 - 0.0001 \\ &\simeq 0.111. \end{aligned}$$

Hence, $x \simeq 3.11$ as found above. But we can find a second approximation for y , using $y = 0.11$. We have

$$\begin{aligned} y &\simeq \frac{2}{17} - \frac{9}{17}(0.11)^2 - \frac{1}{17}(0.11)^3 \\ &\simeq 0.1176 - 0.0064 - 0.0001 \\ &\simeq 0.1111. \end{aligned}$$

Hence,

$$x \simeq 3.111.$$

Similarly, we can find the root of the equation between -3 and -4 .

Trigonometrical equations may also be solved by this method. For example consider the equation

$$10(3 \tan \theta - 2) = \sin \theta,$$

which arises in a certain problem in applied mathematics.

The equation can be written

$$\tan \theta = \frac{2}{3} + \frac{1}{30} \sin \theta.$$

Now $\frac{1}{30} \sin \theta$ never exceeds $\frac{1}{30}$ numerically and hence $\tan \theta \simeq \frac{2}{3}$. A closer approximation is given by substituting in the neglected term $(\frac{1}{30} \sin \theta)$, the value of θ given by $\tan \theta = \frac{2}{3}$, that is, $\sin \theta = \pm 2/\sqrt{13}$.

We note that when $0 < \theta < \frac{1}{2}\pi$, $\tan \theta = \frac{2}{3}$ and $\sin \theta = 2/\sqrt{13}$, but when

$$\pi < \theta < \frac{3}{2}\pi, \quad \tan \theta = \frac{2}{3} \quad \text{and} \quad \sin \theta = -2/\sqrt{13}.$$

Hence we get

$$\begin{aligned} \tan \theta &\simeq \frac{2}{3} \pm \frac{1}{30} \left(\frac{2}{\sqrt{13}} \right) \\ &\simeq 0.6667 \pm 0.0185 = 0.685 \quad \text{or} \quad 0.648. \end{aligned}$$

Hence

$$\theta \simeq 34^\circ 24' \quad \text{or} \quad 180^\circ + 32^\circ 57',$$

restricting ourselves to the solutions between 0° and 360° .

Many other trigonometrical examples might be given, but this must suffice.

No novelty is claimed for these methods, which are well known, but recent experience shows that most pupils and some teachers are not familiar with them. If the latter are stimulated to use them and to examine similar applications in other fields, such as numerical differentiation and integration, the writing of this note will have been well worth while.

J. TOPPING.

THE TRAINING OF THE TEACHER.

BY PROFESSOR T. PERCY NUNN, M.A., D.Sc.

[*Introductory Note.*—Most of those who teach mathematics in secondary schools are in one of three categories :

- (A) Those with an honours degree in mathematics, or a general degree with mathematics as one subject ; they may or may not have had training in the teaching of this subject.
- (B) Those who have no degree but have taken mathematics as a special subject in a course of one or two years at a training college (emergency or otherwise).
- (C) Those who, with or without a degree, have never taken mathematics as a special subject but find themselves committed to teaching it.

All these need or would welcome guidance in three matters :

- (X) Subject—what mathematics is about, as distinct from
- (Y) Applications—an ability to seize on applications of the various processes of mathematics at all stages for purposes of illustration ; this is closely connected with
- (Z) Method—how to adapt the presentation of the subject to the particular groups of pupils for whom the teacher is responsible.

A proper understanding of X should enable the teacher to pick out some of the fundamental notions in the subject. These notions, because they are fundamental, will correspond to something fundamental in the workings of the human mind, and will therefore be related to Z. To concentrate on Z and neglect X produces ventriloquists bringing up ants to talk like parrots.

The need for guidance in these matters was specially brought to the notice of the Teaching Committee through the enthusiasm of men and women who, finding themselves in category C, have voiced their anxiety to teach the subject in the best possible way. There is no doubt however that there are teachers in all three categories who need help with X and Y. Accordingly the Teaching Committee last January set up a subcommittee (called, for want of a better suggestion, the Professional Training subcommittee) to draw up a report for teachers in all three categories and for all who have teachers of mathematics under their charge whether in schoolwork or in training colleges.

This subcommittee (which has held three meetings in 1950) has been reminded of a special number of the *Gazette*—that for December 1919 (vol. IX, no. 143)—which is almost entirely given up to a Report on the Teaching of Mathematics in Public and Secondary Schools. To this Report is attached an Appendix consisting of a few articles expressing not the considered opinions of the Teaching Committee but the personal views of the writers. One of these articles is by the late Percy Nunn, and is entitled "The Training of the Teacher". Our present subcommittee regards this article as so relevant to-day that it has asked for a reprint of it, and this accordingly appears below.

The chief reason for reprinting this article is to obtain the comments of any readers who may be interested, bearing in mind the present significance of "Secondary" which is wider than it was when Nunn was writing. Such comments will be very much appreciated, and should be sent (within a month if possible, but the sooner the better) to the Hon. Secretary of the subcommittee, Mr. M. W. Brown, 62 Fairway, Carshalton Beeches, Surrey. Suggestions for an alternative title for the proposed report will also be welcome. J. T. C.]

CURRENT opinions on the value of training for the teacher cover the widest possible range. There are stalwarts who believe that if a teacher knows "pedagogy" he need know little else ; there are sceptics who think that

nothing matters except knowledge of the subject to be taught, and dismiss the pretensions of the training college as mischievous humbug. There is something to be said for both these extreme views. Just as an ignorant doctor may often, by a "good bedside manner", stimulate the *vis medicatrix naturae* which really cures his patient, so a teacher ignorant of his subject but cunning in his art, may often awaken in the pupil the inner energies that are the ultimate source of educational progress. And just as a ripe orange must yield its juice when squeezed, so the poorest of teachers, if he has learning, must deliver it to pupils who are eager enough to suck it from him.

To apologise for these antithetical opinions is to expose their weakness, and to show that neither can stand unless supplemented by the other. On the one hand, it is futile for a teacher to evoke an "interest" which he cannot feed with the nutriment it craves; on the other hand, learning in a teacher is a vain possession if he cannot persuade his pupils, first, that it is worth having, and secondly, that he can give it to them. Nothing is to be gained by arguing about the relative importance of conditions where both are indispensable. It is plain that the teacher of mathematics must both learn his subjects and learn how to teach them; the only questions worth discussing are questions of ways and means.

Upon one of these questions there is now general agreement among competent judges. In the making of the teacher, the "academic" stage—learning the subject—should be completed before the "professional" stage—learning how to teach it—is seriously begun. There are several reasons for this policy, but the most cogent is that the student's mind ought not to be distressed between two interests of which each should be large enough and active enough to drive all other interests into the background. During his undergraduate years, therefore, the future teacher should be simply a student among other students, conscious that he has a vocation awaiting him, but, in the meantime, throwing himself whole-heartedly into the purely intellectual activities which the pursuit of his subject entails. It follows that the scope and character of his mathematical studies should not be limited by his (supposed) professional needs. This is not to admit that existing University courses in mathematics are plenarily inspired, nor to deny that many of them would be much more useful to teachers if they were somewhat drastically changed. But such changes are to be pressed for on the broad ground that what the teacher needs above all other students is a course which represents adequately the essential genius of mathematics, and that, as things are, he cannot always get it.

Having made this position clear we may pass on to consider, in its broad lines, the student's professional training. Here we must bear in mind that we are concerned not merely to turn out a competent craftsman, but to form a young man or a young woman into an enlightened member of a body that has an enormous responsibility for the wellbeing of a nation. Before the young graduate's outlook has suffered the narrowing so often produced by professional routine, while he is still warm with the generous and universal spirit that University life should have awakened in him, he should be led to inquire into the meaning of education and to understand something of its significance in relation to the many-sided business of life. He should learn how much wider education is than mere teaching, and should gain inspiration and right direction from those who have reflected most deeply upon it. And he should have his bias as a specialist corrected by observing how all the major subjects of the curriculum answer to deep-seated needs of the human spirit and represent essential currents of the great stream of movement called civilisation.

Nothing could be, in the long run, more unwise than to exclude from the

professional course those elements of breadth and liberality in which, as everyone agrees, the chief virtue of academic studies resides. Nevertheless the course will miss its point unless it also includes an adequate training in the teaching-craft that belongs to the student's special subject. It is not sufficient for him to learn the common arts of exposition and class-management; he must learn what forms those arts assume when applied, for example, in the field of mathematical teaching. The first thing he should discover here is that the art of teaching does not consist in the mastery of a number of tricks by which young minds may be persuaded to accept a mass of ready-made knowledge; but that it is a process whose aim is to guide the pupils' mental activities along a path of development in which he repeats and makes his own some of the great historic achievements of the science. In other words, he must learn that to teach mathematics is not merely to seize the pupil of certain knowledge in arithmetic, algebra, geometry and the rest, but to make him, in the greatest measure possible, an active intellectual adventurer in the realms of number and space, following up the traces of the great masters of mathematical thought and catching something of their spirit and outlook. To make the best use of this discovery the student must work again over the familiar field of elementary mathematics, studying it, this time, not in the naive attitude of a learner, but as a critic interested in finding out whence mathematical thought springs, how it develops and whither it leads. In this part of his training he should be taught to question the accepted values, and to inquire in a critical spirit what parts of the traditional curriculum are really vital and what parts have only a conventional value. He should also be led to study the reactions to mathematical teaching of minds differing in age and natural bent: to observe the appeal, in varying circumstances of the intrinsic beauty, the utility and the logical unity of mathematical truths.

The question who should teach the teacher his trade must be regarded as still unsettled. Some hold that he should learn his craft entirely in a school under the eye of a master-craftsman; others that he should spend his year of training under the direct influence and guidance of a training college or pedagogical department of a University. The former plan has the obvious merit of keeping the young teacher constantly in touch with the actual conditions of his profession, but it has serious drawbacks. A school can rarely be a place where a student can gain that philosophic outlook upon education as a whole upon which we have insisted, or study effectively the more recent contributions of psychology to the business of teaching. Moreover, there is only a very small number of highly competent teachers of mathematics who can spare the time and have the special interest needed both to supervise adequately the aspirant's technical progress and to guide his studies in what we have spoken of as the criticism of his subject. The ideal solution of the problem of training would seem to involve an institution of University rank—a centre of educational criticism and inspiration, and a clearing-house of educational ideas—with schools working in such close relations with it as to be true organs of its spirit. It is, perhaps, not extravagant to hope that as our newer Universities gather strength, such institutions will grow up in them, and become, so to speak, the centres of consciousness of the teaching profession, each in its own province. Having earned the necessary prestige, they could not only supervise the apprenticeship of young teachers, but also perform the functions of the military "staff college" as places to which eminent teachers could be invited to withdraw for a while from the busy routine of school life in order to give their colleagues in the schools of the province the benefit of their experience and special skill, and to keep them well in the stream of progress. For this, too, is an aspect of the question of training of no less importance than the one which the term ordinarily suggests.

MATHEMATICAL NOTES.

2188. "Surprising."

In Note 2091 it is said to be surprising that in the first 700 figures of π , as calculated by Shanks, 7 occurs as seldom as 51 times, 19 below expectation. I get the odds against some figure occurring 51 times or less to only 12 to 1; and if this is right it can hardly be called a wonderful coincidence. But such as it is, it is equally wonderful whether it occurs in the wrong figures obtained by Shanks or in the right ones calculated by Ferguson: both are "random" figures unless one assumes Shanks's errors to have been dictated by some anti-seven bias. "Random", of course, does not mean determined by no system at all, but by no system which, in relation to the knowledge possessed by the reader or even the computer, makes one figure likelier than another before it is known.

Much more remarkable is the occurrence of six consecutive 9's in the first 800 figures. W. HOPE-JONES.

2189. *The digits in the decimal form of π .*

The table in Note 2091 (XXXIII, p. 291) must not be thought to dispose of a very real problem. If in professing to write down a value of $\sqrt{3}$ I interchange the second and third decimals, what I produce is not an accidental decimal but a value of $\sqrt{3} - 0.009$; if intending to find $\sin 1^\circ$ from the series I forget the third term, I am evaluating the definite number

$$\sin 1^\circ - \pi^3/2^{13} \cdot 3^{11} \cdot 5^6.$$

If the sources of all of Shanks' mistakes could be identified, we could say that although he thought he was evaluating π , he was in fact evaluating a different number $\pi + 10^{-530}S$. Is a deficiency of 7's in this number less queer than a deficiency in π would have been? Surely the fact is that we were ready to believe anything of π , but that with regard to this other quite uninteresting number we have no doubt that any peculiarity is apparent rather than real.

The question is, what is the rule and what is the exception in this matter? Striking though a proved case of digital excess or deficiency would be, have we any ground for expecting the proportions of the several digits to tend to fixed limits, let alone to tend to the same limit? Two considerations render me sceptical.

In the first place, digital irregularity is primarily relative to a scale of notation. If the digital frequency tends to regularity in one scale, it is difficult not to suppose that it will tend to regularity in any other. But even if the digital frequency is not statistically regular in one scale, if the distribution of each individual digit is in any sense a random distribution, we should expect a change of radix to distribute the irregularities regularly and to produce a regular ultimate frequency in the new scale. Returning from the second scale to the first we have a paradox to remind us that nothing in mathematics is more futile than conjecture.

My second reason for doubt is more fundamental. A dogma which we assert when we have in mind the familiar decimal scale must bear assertion in any other scale of notation. We say we are "sure" that the decimal form of π "ought" to have in the long run a due proportion of 7's. Are we just as "sure" that it "ought" to include a due proportion of runs of a million million 7's in succession? Yet this is a requirement less drastic than the regular occurrence of an assigned digit in the scale whose radix is 10 raised to the power of 10^{12} , and we can not maintain that the validity of intuition depends on the size of the radix. When we let the petty scales of everyday

arithmetic fall from our eyes, we see that we still have a great deal to learn from the immortal monkey with a typewriter.

E. H. N.

2190. Every conic consists of two straight lines.

Let S be any given conic. On it take two points Y, Z to be the vertices of a triangle of reference for general homogeneous coordinates, the third vertex X being arbitrary. The equation of the conic then assumes the form

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

where

$$b = c = 0.$$

The equation in line coordinates of the point-pair (Y, Z) is

$$\Sigma' \equiv 2mn = 0;$$

the coefficients in the corresponding point equation $S' = 0$ are the respective cofactors in the determinant

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}.$$

Hence

$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0,$$

where

$$a' = -1,$$

$$b' = c' = f' = g' = h' = 0.$$

The discriminant of the pencil $S + \lambda S' = 0$ is

$$\begin{vmatrix} a - \lambda & h & g \\ h & 0 & f \\ g & f & 0 \end{vmatrix},$$

or

$$2fgh - (a - \lambda)f^2,$$

or

$$(2fgh - af^2) + \lambda f^2.$$

In the standard notation of invariants, the discriminant is

$$\Delta + \Theta\lambda + \Theta'\lambda^2 + \Delta'\lambda^3,$$

and so, comparing the two forms, we obtain the relations

$$\Theta' = 0, \quad \Delta' = 0.$$

The relation $\Delta' = 0$ may be dismissed at once; it merely expresses the fact that S' is degenerate.

Consider, then, the relation

$$\Theta' = 0,$$

or, by the well-known formula,

$$aA' + bB' + cC' + 2fF' + 2gG' + 2hH' = 0,$$

where A', B', C', F', G', H' are the coefficients in the equation $\Sigma' = 0$. Remembering that $\Sigma' \equiv 2mn$, we have

$$F' = 1,$$

$$A' = B' = C' = G' = H' = 0.$$

The relation therefore reduces to

$$f = 0,$$

and so the equation of the given conic S is

$$ax^2 + 2gzx + 2hxy = 0.$$

Hence the conic S is degenerate, breaking up into the two straight lines

$$x=0, \quad ax+hy+gz=0.$$

E. A. MAXWELL.

2191. *A question in statics.*

"Two equal uniform rods AB, BC are freely hinged at B . C rests on a rough horizontal plane and A is attached to a point above it. When C is as far as possible from A for equilibrium, AB, BC makes angles α, β with the vertical. Prove that the coefficient of friction between the rod at C and the plane is $2/(\cot \alpha - 3 \cot \beta)$."—A. S. Ramsey, *Statics* (Cambridge, 2nd edition), p. 157, Ex. 38.

The straightforward solution is as follows. Let F be the horizontal friction at C (acting outwards, that is, away from the vertical through A) and R the normal reaction at C . Introducing the horizontal and vertical reactions at B , we get one equation for AB , by moments about A , and three equations for BC , whence easily $F/R = 2/(\cot \alpha - 3 \cot \beta)$.

But this is not the whole story. Let the length of each rod be unity and the height of A above the plane be h . Of course, $h < 2$ and we may take in the first place $h > 1$. The angle α can vary from 0 to $\cos^{-1}(h-1)$ ($=\alpha_1$, say), and β , since $\cos \alpha + \cos \beta = h$, varies from α_1 to 0. Let α increase from 0 to α_1 , the coefficient of friction being μ . Then F/R is zero when $\alpha=0$, increases as α increases, becoming μ for $\alpha=\alpha_1$ (say) and ∞ when $\alpha=\alpha_2$ (say). Then F/R becomes negative, increasing from $-\infty$ to 0 as α increases from α_2 to α_3 , and passing through the value $-\mu$ at α_3 (say). So there is equilibrium when $0 \leq \alpha \leq \alpha_1$ or $\alpha_3 \leq \alpha \leq \alpha_4$. In the second range the friction, of course, acts inwards. It is easily shown that the value of α (α' , say) which makes ABC a straight line ($\alpha' = \cos^{-1} \frac{1}{2}h$) is greater than α_2 and may be greater or less than α_3 . In the first case, α' obviously gives the maximum value of CD (where D is the point of the plane vertically below A).^{*} In the second case, the maximum value of CD is either at α_1 or α_3 , and it can be shown, in fact, that it is always at α_3 . It can be shown also that the first case occurs if

$$\mu > \sqrt{(4/h^2 - 1)}.$$

A slight modification is necessary if $h < 1$.

A related problem is when the two rods slope in opposite directions (it is tacitly assumed above that they slope in the same direction). It will be found that F/R has a finite maximum in this case, which may or may not be greater than μ .

F. G. MAUNSELL.

2192. *Similar right-angled triangles.*

The proposition that two right-angled triangles are similar if the ratio of one side to the hypotenuse is the same in each triangle follows as a direct deduction from Euclid, Book VI, Prop. 7, which runs (Hall and Stevens, *Euclid*, 1898, p. 321): "If two triangles have one angle of the one equal to one angle of the other and the sides about one other angle in each proportional, so that the sides opposite to the equal angles are homologous, then the third angles are either equal or supplementary; and in the former case the triangles are similar." But as the first-mentioned proposition is the geometrical proof of the uniqueness of the acute angle which has a given sine or cosine, corresponds to the case of congruent right-angled triangles and does not appear in many geometry textbooks, it seems worth giving some direct proofs.

Let the particular enunciation be: in the triangles ABC, DEF the angles

^{*} A solution with $\alpha=\alpha'$ is physically absurd, as the reaction becomes infinite, but α can be as near α' as we wish.

ABC, DEF are each one right angle, and $AB : DE = AC : DF$; prove that the triangles are similar.

Proofs depending on congruent right-angled triangles are (i) construct on AB , on the side away from C , a triangle similar to the triangle DEF , and prove its congruence with the triangle ABC ; (ii) cut off from AB (assumed greater than DE) a length AE' equal to DE , draw through E' a line parallel to BC cutting AC at F' , and prove that the triangle $AE'F'$ is congruent with the triangle DEF .

The use of the Pythagoras theorem and ratios, which is equivalent to proving that if $\sin x$ or $\cos x$ is given, $\tan x$ is fixed for an acute angle, seems hardly geometry.

I am indebted to a candidate in a scholarship examination for the following direct proof. The middle points M of AC , N of DF are the centres of the circles ABC, DEF , and therefore in the triangles AMB, DNE we have

$$AB : DE = BM : EN = MA : ND,$$

and these triangles are similar and the angles $BAM(C), EDN(F)$ are equal. Therefore the triangles ABC, DEF are similar, two angles of the one being equal to the two corresponding angles of the other.

J. J. WELCH.

2193. The remainder theorem for operators.

The controversy about the remainder theorem suggested the following note.

The division process is possible even for non-commutative variables. For instance, if D is the operator d/dx , and $f(D)$ stand for an operator of Landsberg's type,

$$f(D) \equiv \sum_0^n F_k(x) D^k,$$

there is a unique quotient $q(D)$ of the same type $f(D)$ and a unique remainder $R(x)$, such that

$$f(D) \equiv q(D) \cdot (D - z) + R(x), \dots\dots\dots(i)$$

where z is any given function of x , so that we can speak of dividing $f(D)$ by $D - z$. Let $v = \int z dx$. Then

$$(D - z) \cdot e^v = 0. \dots\dots\dots(ii)$$

Let the two sides of identity (i) operate on e^v . It follows that

$$f(D) e^v = R(x) e^v. \dots\dots\dots(iii)$$

The left-hand side of (iii) can easily be shown to be equal to $e^v \cdot f(D + v') \cdot 1$, where

$$f(D + v') \equiv \sum_0^n F_k(x) (D + v')^k.$$

Hence, as $v' = z$,

$$R(x) = f(D + z) \cdot 1.$$

If z is simply a constant, this result reduces to $R = f(z)$. The interesting point is that D is not a constant at all; the question of "putting D equal to z " does not arise.

This result may be well-known. Does anyone recognise it?

W. W. SAWYER.

2194. Skew-symmetric determinants of even order.

In Note 2093 Dr. Busbridge gave a proof of the theorem that a skew-symmetric determinant of even order is the square of a polynomial in its elements. I think the following proof is more economical, and no less acceptable to first-year students.

Suppose that the determinant is $|a_j^i|$, and that at least one element in the first row, say a_m^1 , is not zero. If another element in this row, a_j^1 , is not zero, multiply the m th column by a_j^1/a_m^1 and subtract from the j th column: this replaces a_j^1 by zero without affecting the m th row (since $a_m^m = 0$) or the value of the determinant. In this way every element in the first row except the m th can be replaced by zero. In a similar way, all the elements of the m th row except the first can now be replaced by zero without affecting the new first row. The elements b_j^i of the new determinant are given by

$$b_j^i = a_j^i - \frac{a_j^1}{a_m^1} a_m^i - \frac{a_j^m}{a_1^m} a_1^i = -b_i^j,$$

except when j is 1 or m . Hence the determinant D obtained by deleting the first and m th rows and columns is skew-symmetric. But $|a_j^i| = (a_m^1)^2 D$, whence the theorem follows by induction. J. D. WESTON.

2195. Orthogonality of Bessel functions.

One immediate consequence of the well known orthogonality relations

$$(1) \quad \int_0^1 J_n(\mu x) J_n(\nu x) x dx = \begin{cases} 0 & (\mu \neq \nu), \\ \frac{1}{2} [J_n'(\mu)]^2 & (\mu = \nu), \end{cases}$$

where μ and ν are zeros of the Bessel function $J_n(x)$, is the fact that the kinetic energy of a vibrating circular membrane is the sum of the kinetic energies in the separate normal modes of which the vibration is compounded. For a direct proof of the corresponding fact for the potential energy, one needs the relations

$$(2) \quad \int_0^1 \left\{ x \frac{d}{dx} J_n(\mu x) \frac{d}{dx} J_n(\nu x) + \frac{n^2}{x} J_n(\mu x) J_n(\nu x) \right\} dx = \begin{cases} 0 & (\mu \neq \nu), \\ \frac{1}{2} \mu^2 [J_n'(\mu)]^2 & (\mu = \nu). \end{cases}$$

These can, of course, be deduced from (1) by means of recurrence formulae, since (2) is equivalent to

$$\int_0^1 F_n(\mu x) F_n(\nu x) x dx = \begin{cases} 0 & (\mu \neq \nu), \\ \frac{1}{2} \mu^2 [J_n'(\mu)]^2 & (\mu = \nu), \end{cases}$$

where

$$F_n(\mu x) = \frac{d}{dx} J_n(\mu x) + \frac{n}{x} J_n(\mu x).$$

However, (2) can be obtained directly from the differential equation for $J_n(\mu x)$ in much the same way as (1) is usually obtained. We have

$$x J_n(\nu x) \left\{ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \mu^2 - \frac{n^2}{x^2} \right\} J_n(\mu x) = 0.$$

Interchanging μ and ν and adding, we get

$$\begin{aligned} & 2 \left\{ x \frac{d}{dx} J_n(\mu x) \frac{d}{dx} J_n(\nu x) + \frac{n^2}{x} J_n(\mu x) J_n(\nu x) \right\} \\ & = (\mu^2 + \nu^2) x J_n(\mu x) J_n(\nu x) + \frac{d}{dx} x \left\{ J_n(\mu x) \frac{d}{dx} J_n(\nu x) + J_n(\nu x) \frac{d}{dx} J_n(\mu x) \right\}. \end{aligned}$$

Integration, taking (1) into account, now yields (2).

J. D. WESTON.

2196. A peculiar function arising from an everyday problem.

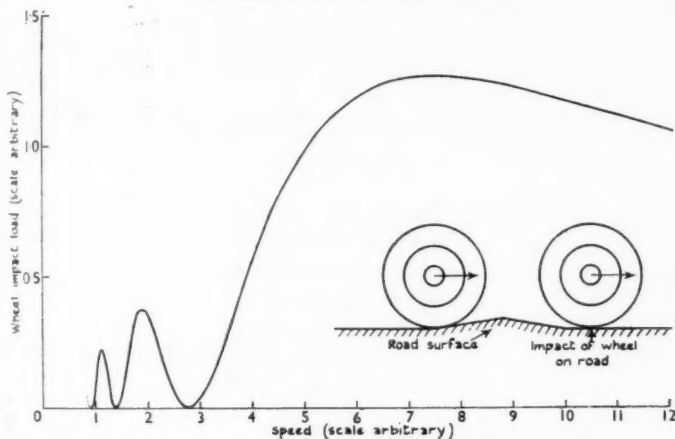
In so far as vertical movement is concerned, a wheel on a tyre may be represented as a concentrated mass m supported on a weightless spring of stiffness λ ; crude as the model may appear, it affords a surprisingly good representation of actual wheels. If this wheel rolling with velocity V along

a flat surface encounters an inclined plane of slope α , the increase of wheel load at time t afterwards is $(\lambda V \alpha / p) \sin pt$, where $p^2 = \lambda / m$ and p is the natural frequency of the wheel on its tyre. If at some later time t_0 the wheel encounters another change of slope β , so that the new plane of travel is inclined upwards at the slope $\alpha + \beta$, the wheel load at time $t > t_0$ is

$$(\lambda V / p) \{ \alpha \sin pt + \beta \sin p(t - t_0) \},$$

and so on for subsequent changes of slope.

Variation of Wheel Impact Load with Speed



Now, owing to local deformation of the tyre tread, the wheel tends to "ride" over any abrupt change of level in the road surface. As a result no obstacle causes more than a change of slope of "effective road surface", and an obstacle such as a plank of moderate width becomes in effect an isosceles triangle. If, then, the wheel encounters an obstacle in the form of an isosceles triangle of height h and length $2a$, the wheel load after the complete obstacle has been traversed is

$$(\lambda V h / pa) \{ \sin pt - 2 \sin p(t - a/V) + \sin p(t - 2a/V) \},$$

the changes of slope being $+(h/a)$ at $t=0$, $-(2h/a)$ at $t=(a/V)$ and $+(h/a)$ at $t=(2a/V)$. On simplification this wheel load due to "impact" becomes $-(2\lambda V h / pa) \{ 1 - \cos(pa/V) \} \sin p(t - a/V)$, and the maximum value of this impact of the wheel on the road behind the obstacle is

$$2\lambda h \{ 1 - \cos(pa/V) \} / (pa/V).$$

Now the function $(1 - \cos x)/x$ is, of course, quite ordinary; it is merely periodic with a sequence of zero values between a sequence of maxima which decrease as x increases. On the other hand, in the wheel impact case the current variable x is represented by pa/V , and the ratio of wheel load to $2\lambda h$ plotted against V/pa is the same as $(1 - \cos x)/x$ plotted against $1/x$. The resulting curve looks unfamiliar, and experimental readings of impact load which do indeed conform quite closely to this curve might be considered unreliable if the theoretical basis were not understood.

H. L. Cox.

2197. *A fallacy.*

It is well known that the surface area of a belt of a sphere is equal to the surface of the corresponding belt of the circumscribing cylinder. It follows that on the earth's surface the surface area of a polar cap of depth h is equal to the area of an equatorial belt of the same depth. The equatorial belt, however, clearly has the larger volume.

Now, the radiating surfaces being equal, the greater volume will cool more slowly, which explains why the earth is warmer at the equator than at the poles.

GEORGE TYSON.

2198. *The triple vector product.*

In Note 1109 of the *Gazette* for July 1934 I gave a proof of the formula for the triple vector product, which although short can be shortened still further by using Milne's notation $\mathbf{P}_Q, \mathbf{P}_Q^*$ for the components of \mathbf{P} along \mathbf{Q} and at right angles to \mathbf{Q} in the plane of \mathbf{P} and \mathbf{Q} .

First we note that, if \mathbf{F} is a vector perpendicular to \mathbf{C} , then \mathbf{C}, \mathbf{F} and $\mathbf{C} \wedge \mathbf{F}$ form an orthogonal triad and hence it follows easily that

$$(\mathbf{F} \wedge \mathbf{C}) \wedge \mathbf{F} = F^2 \mathbf{C},$$

and hence that

$$(\mathbf{B}'_C \wedge \mathbf{C}) \wedge \mathbf{B}'_C = B'^2_C \mathbf{C}. \dots\dots\dots(i)$$

Now

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = \lambda \mathbf{B} + \mu \mathbf{C},$$

where λ, μ are scalars. Multiplying scalarly by \mathbf{B}'_C ,

$$\lambda (\mathbf{B}'_C \cdot \mathbf{B}) = \mathbf{B}'_C \cdot \mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}).$$

Thus

$$\lambda (\mathbf{B}'_C \cdot \mathbf{B} + \mathbf{B}'_C \cdot \mathbf{C}) = \mathbf{A} \cdot (\mathbf{B} \wedge \mathbf{C}) \wedge \mathbf{B}'_C.$$

so that

$$\begin{aligned} \lambda B'^2_C &= \mathbf{A} \cdot \{ (\mathbf{B}_C + \mathbf{B}'_C) \wedge \mathbf{C} \} \wedge \mathbf{B}'_C \\ &= \mathbf{A} \cdot (\mathbf{B}'_C \wedge \mathbf{C}) \wedge \mathbf{B}'_C \\ &= \mathbf{A} \cdot B'^2_C \mathbf{C}, \quad \text{by (i).} \end{aligned}$$

Hence $\lambda = \mathbf{A} \cdot \mathbf{C}$, and similarly $\mu = -(\mathbf{B} \cdot \mathbf{C})$.

H. V. LOWRY.

2199. *On energy of impact.*

In a reconsideration of this question (incidental to a wider context of energy) the following details, which may be of interest, have emerged. It is sufficient to consider the case of direct impact of two "particles" P_1, P_2 .

1. Using obvious notation, the equations for determination of v_1, v_2 from given u_1, u_2 are

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

or

$$m_1 (u_1 - v_1) = m_2 (v_2 - u_2) \dots\dots\dots(ii)$$

and

$$v_2 - v_1 = e(u_1 - u_2), \dots\dots\dots(iii)$$

in which u_1 and $(u_1 - u_2)$ are positive and $0 \leq e \leq 1$.

2. The point of the method is to take first the important special case in which $e = 1$. Equation (ii) may then be put as

$$u_1 + v_1 = v_2 + u_2,$$

and thence multiplying into equation (i),

$$m_1 (u_1^2 - v_1^2) = m (v_2^2 - u_2^2)$$

or

$$m_1 v_1^2 + m_2 v_2^2 = m_1 u_1^2 + m_2 u_2^2;$$

that is, in this case, conservation of energy of motion.

3. With § 2 in mind, we "solve" the simultaneous equations (i), (ii) in the form

$$m_1(m_1 + m_2)(u_1 - v_1) = m_2(m_1 + m_2)(v_2 - u_2) \\ = (1 + e)m_1m_2(u_1 - u_2) \dots\dots\dots(iii)$$

(proving algebraically that $(u_1 - v_1)$ and $(v_2 - u_1)$ are positive—which are intuitively evident facts). Then, writing (ii) in the form

$$(v_2 + u_2) - (u_1 + v_1) = (e - 1)(u_1 - u_2)$$

or

$$(v_2 + u_2) = (u_1 + v_1) - (1 - e)(u_1 - u_2),$$

and multiplying the respective terms by the three corresponding equal expressions of (iii),

$$m_2(v_2^2 - u_2^2) = m_1(u_1^2 - v_1^2) - (1 - e^2)\mu(u_1 - u_2)^2$$

or

$$m_1v_1^2 + m_2v_2^2 = m_1u_1^2 + m_2u_2^2 - (1 - e^2)\mu(u_1 - u_2)^2,$$

where

$$\mu = m_1m_2/(m_1 + m_2):$$

the standard theorem on loss of energy of motion due to the impact.

4. The direct algebraic treatment of this elementary problem has a certain fundamental significance which cannot be discussed in this Note.

D. K. PICKEN.

2200. Loss of energy on impact.

A possibly more significant approach to the familiar formula given by Mr. W. H. E. Bentley in Note 2059 for the loss of energy of two masses in direct collision, namely,

$$\frac{1}{2} \frac{m_1m_2}{m_1 + m_2} (1 - e^2)(u_1 - u_2)^2,$$

is by calculating the difference between the actual energy and the "energy of the centre of gravity", that is, the energy which would be possessed by a particle of mass $m_1 + m_2$ moving with the velocity of the centre of gravity. This difference is

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1 + m_2) \left(\frac{m_1u_1 + m_2u_2}{m_1 + m_2} \right)^2 = \frac{1}{2}\mu(u_1 - u_2)^2,$$

where

$$\mu = m_1m_2/(m_1 + m_2).$$

This term may be called the "relative energy". On collision, the velocity of the centre of gravity is unaffected and the relative velocity is multiplied by $-e$. Hence the loss of energy on collision is equal to the decrease of "relative energy"

$$= \frac{1}{2}\mu(1 - e^2)(u_1 - u_2)^2.$$

Using the above analysis the result may be immediately extended to oblique collision, the result being given by the same expression, where u_1, u_2 now represent the components of velocity along the line of centres at the instant of contact. For, the energy of a moving body is the sum of the energies of velocity components in two perpendicular directions, and since the components perpendicular to the line of centres are both unaltered, the only loss of energy is that given by the above formula. F. M. GOLDNER.

1664. Among masters who are also friends I must assign a high place to the Rev. William Done Bushell, who vainly endeavoured to teach me mathematics, but found me more at home in the sphere (which he also loved) of Ecclesiology.—G. W. E. Russell, *Fifteen Chapters of Autobiography*, Ch. II, "Harrow". The Rev. W. D. Bushell was one of the founders of our Association; his son, Mr. W. F. Bushell, was President in 1946.

REVIEWS.

Zahlentheorie. By H. HASSE. Pp. xii, 468. DM. 44. 1949. (Akademie-Verlag, Berlin)

Though there seems no end to books on the Theory of Numbers, the subject is so flourishing that there has been need for many more. Perhaps the need has been greatest in those regions associated with the algebraic theory of numbers. The extension by Gauss of the laws of arithmetic of the rational numbers to the complex numbers of the form $x + iy$ where x, y are rational numbers presented no special difficulty in the early part of the last century. It was otherwise, however, when more general numbers were considered.

Let θ be the root of an equation

$$f(\theta) = a_0\theta^n + a_1\theta^{n-1} + \dots + a_n = 0,$$

where a_0, a_1, \dots, a_n are rational integers, say, integers in the rational field P , the existence of a root being, of course, really a question in Analysis, and suppose that $f(\theta)$ is irreducible in P . Then the rational functions of θ with coefficients in P ,

$$\phi(\theta) = A(\theta)/B(\theta),$$

where $A(\theta), B(\theta)$ are polynomials in θ with coefficients in P , form a field, say $P(\theta)$. To develop an arithmetic in $P(\theta)$, we must first pick out the ring of the integer elements in $P(\theta)$, that is, the elements which form a closed set under addition and multiplication and contain the number 1. These, of course, are not given *a priori* so simply as happens with the Gaussian complex integers. We must then examine whether the usual laws of arithmetic apply, for example, to factorisation. It is soon seen that the obviously suggested definition of prime numbers is no longer useful, since this does not lead to a unique factorisation theorem in terms of the primes.

The difficulty first arose in Kummer's study of the field $P(\zeta)$ where ζ is a complex p th root of unity, $\zeta^p = 1$, and p is a prime. The obvious definition of an integer suffices, namely, an element ϕ of the field is an integer if

$$\phi = x_0 + x_1\zeta + \dots + x_{p-2}\zeta^{p-2},$$

where x_0, x_1, \dots, x_{p-2} are integers in P . This is also equivalent to saying that ϕ is an integer if it satisfies an equation irreducible in P ,

$$\phi^r + b_1\phi^{r-1} + \dots + b_r = 0,$$

where the b 's are integers in P . The question of factorisation was completely solved by Kummer. The difficulty for the more general field $P(\theta)$ or $\Omega(\theta)$ where Ω is an arbitrary field is essentially the same as for the cyclotomic field $P(\zeta)$, namely, that the number field $P(\theta)$ is not sufficiently wide to permit of unique factorisation within it. However, by considering an extended set of new numbers P' , including not only $P(\theta)$ but also the numbers $\sqrt[h]{S(\theta)}$ where $S(\theta)$ belongs to a special subset of $P(\theta)$ and h is a number depending only on the field $P(\theta)$, all the methods developed show that unique factorisation holds for the number of P' in terms of the prime elements of P' .

More generally, we consider a set of A of elements A_1, A_2, \dots related to the field $P(\theta)$, and of various kinds, e.g. ideal numbers, ideals, divisors, functionals, not necessarily numbers. For these, prime elements p_1, p_2, \dots and multiplication are defined and are such as to permit of unique factorisation of the A 's. Also the A 's contain subsets isomorphic to the integers of $P(\theta)$. Then an element Φ of A can be factorised as

$$\Phi = p_1^{\lambda_1} \dots p_r^{\lambda_r},$$

and if Φ isomorphic to ϕ in $P(\theta)$, we can indicate this factorisation by

$$\phi \cong p_1^{\lambda_1} \dots p_r^{\lambda_r}$$

and consider $p_1^{\lambda_1}$, etc., as the factors of ϕ , though of course these need not be elements of $P(\theta)$. The various ways of treating the subject will depend upon the definition of the A 's and the method of introducing the factorisation. Two essentially different methods of doing this have been used, one based on divisors and the other on ideals. The first arose from the arithmetical investigations of Kummer and Kronecker, and the utilisation of Weierstrass' ideas in function theory. It was further developed by Hensel and new foundations were laid in the general theory of fields by Steinitz and in the general valuation theory of Kurschak, Ostrowski and others. The ideal theory was due to Dedekind, improved by Hilbert and further studied and developed by Emmy Noether, Artin and others.

To resume the earlier history, Kummer in his study of the field $P(\zeta)$ introduced the A 's as ideal numbers, that is, no explicit expression was used to define them. The criterion of divisibility of a number ϕ by an ideal prime \mathfrak{p} was the solvability of certain congruences in P . Then Dedekind treated the more general case of the field $P(\theta)$ by considering the A 's as ideals, that is, the sets of numbers obtained from

$$x_1\theta_1 + \dots + x_r\theta_r,$$

where the θ are given and the x 's take all integer values in P . Kronecker considered the A 's as functionals, that is, the elements A were now rational functions of variables x, y, z, \dots with coefficients in $P(\theta)$ where, however, the x, y, z, \dots served as symbols. In fact, the functionals define function fields $P(\theta, x, y, z, \dots)$ over $P(\theta)$. Weyl in his recent book on algebraic numbers develops the subject mostly in Kronecker's manner.

It was soon realised that the ideas thus introduced into number theory had wider applications. The field $P(\theta)$ is merely a particular case in the concept of more general fields. Then Dedekind and Weber made a detailed arithmetical study of the algebraic functions of a single variable, really the field $\Omega(w, z)$ where w, z are connected by a polynomial equation with coefficients in the field Ω of complex numbers. This and the related results such as the genus and the Riemann-Roch theorem had long been the preserve of the geometers and analysts. It was the idea used here that was to lead Hensel later to his p -adic developments in number-theory. The whole subject has been completely transformed in the last forty years.

In recent years, the study of simple arithmetical questions such as the number of solutions of a congruence

$$f(x, y) \equiv 0 \pmod{p},$$

where p is a prime and $f(x, y)$ is a polynomial with integer coefficients in P has led to the most unexpected consequences and developments. The problem was found to be intimately related to the theory of algebraic function fields. But now the coefficients are no longer complex numbers in Ω , but numbers in other fields, for example, the residues mod p forming a Galois field. A study must be made of the more general case when the coefficients are elements of a given field. Though the theory runs parallel in many respects to that of the field of algebraic numbers, there are important differences. A unified account of all this has long been a desideratum. At last this has been fulfilled in the appearance of Hasse's book.

Here is a beautiful book which will delight the hearts of all those sufficiently interested in number-theory to wish to penetrate into many of its mysteries. It treats the subject from a point of view very different from that of other

writers, and contains a great body of work which has never appeared in a textbook. Its object is to discuss not only the elementary theory of numbers considered as the arithmetic in the field P of rational numbers, but also to discuss the arithmetic of more general fields K and to derive its results from that of a subfield Ω of K . The author develops his subject in such a way and gives proofs of such a type that they can be extended and generalised, the simpler cases then serving as a model for the more general ones.

The book consists of three sections: the rudiments of the arithmetic in the rational number field; the theory of fields with a valuation; and the foundation of the arithmetic in algebraic fields.

The first section deals with what may be called the elementary theory of numbers. Here we are given immediately the ring Γ of rational integers, so that Γ is a closed set under the operations of addition, subtraction and multiplication. From these integers is derived the field P of rational numbers, and in fact P is the quotient field of the ring Γ . The author's aim is to develop also an arithmetic holding for more general fields K . But now the field K is a more obvious set of elements and starting point than its integer domain I with quotient field K , and so I will have to be defined. It becomes important to consider the additive group K^+ and the multiplicative group K^* of the elements of K in order to discuss the ring I . This object is the chief feature of the treatment in Section 1, which is presented in such a form as to be capable of extension and generalisation to more general fields K . In particular, the author is also interested in this section in the field $\Omega(t)$ of the rational functions of an indeterminate t and its integer ring $\Omega[t]$ where Ω is an arbitrary field.

Hasse develops the usual properties of primes and congruences. The residue classes mod m are expressed by means of rings and decomposed into direct sums and products according as the additive or multiplicative aspect of P is considered. The theory of quadratic residues is related to the study of the characters of finite Abelian groups, and then its connection with Hilbert's norm residue symbol is discussed.

The whole book is dominated by the second section. To see what this means, let a be any rational number in the rational field P , and write

$$f(a) = |a|.$$

The function $f(a)$ possesses the properties

$$f(0) = 0, \quad f(a) > 0 \text{ for } a \neq 0,$$

$$f(ab) = f(a)f(b), \quad f(a+b) \leq f(a) + f(b);$$

and any function with these properties may be called a valuation of the field P . But the field P possesses other valuations, for example, the trivial valuation $f(0) = 0, f(a) = 1$ for $a \neq 0$. Again, let p be a prime number and let α_p be the greatest number such that $a/p^{\alpha_p} = r/s$, where r, s are integers both prime to p . Then it is easily seen that a valuation of a is given by

$$|a|_p = p^{-\alpha_p},$$

and indeed there are an infinity corresponding to the infinity of primes p . All of these are discrete valuations and are of fundamental importance. It is then usually more convenient to define a new function of a by

$$w_p(a) = \alpha_p.$$

The valuation $|a|$ for P carries over at once to many other fields containing P as a subfield; for example, the field P_∞ of real numbers has a valuation $g(a) = |a|$ which when a is in P becomes the valuation $f(a)$; again, the field Ω of real and complex numbers has the valuation $|a|$ with now the usual

definition of the modulus of a complex number. Since elementary number theory is largely concerned with primes p , it might be expected that a satisfactory theory could be developed by considering the modified valuation $w_p(a)$. Then the arithmetic in fields K' containing a given field K as a subfield might be expected to follow from the continuation of the valuation of K into K' . The study of convergent sequences of rational numbers where convergence is defined in respect of the modulus, that is $|a_r - a_s| < \epsilon$, leads to the field of real numbers. But now convergent sequences of elements of a field K may be defined with respect to a valuation, say $f(a)$, and lead to the study of the perfect fields over K . In particular, when $K = P$ and $f(a) = p^{-\alpha}$ we have the p -adic number and the p -adic number field P_p with which Hensel enriched number-theory, and which have been a characteristic feature of Hasse's work. Of course, some of this is to be found in books on modern algebra, but it is important for arithmeticians to have an extended account of the theory of the structure of discrete valued perfect fields and in the form in which it will be of most use for arithmetical application.

The third section is concerned with the arithmetic in algebraic extensions of both number and function fields. First the arithmetic in the field P where now P is either the rational field or the field of the rational functions of an indeterminate t over a finite field Ω , is given in terms of the valuations of P . It is then shown that if similar arithmetical properties hold for any field K and its discrete valuations, they also hold for any finite algebraic extension of K . In particular, when K is the rational function field, the distinction between separable and inseparable fields becomes important.

Consider the case of a finite algebraic number field K . Each discrete valuation w leads to the elements a of K that can be called integral with respect to K ; these satisfy $w(a) \geq 0$. The elements a with $w(a) > 0$ define an ideal in K , and by normalising the valuation w , we can find elements π with $w(\pi) = 1$. These elements π we call prime elements, and all such elements π can be called prime elements. The set of the π can be denoted by \mathfrak{p} , and then in all divisibility and congruence conditions a power π^α can be replaced by \mathfrak{p}^α . We now write w_p for the valuation w and then the statement $w_p(a) = \alpha$ can be written as

$$a \cong \mathfrak{p}^\alpha.$$

We now introduce our divisor

$$a = \prod_{\mathfrak{p}} \mathfrak{p}^{\alpha_{\mathfrak{p}}},$$

where $\alpha_{\mathfrak{p}} = 0$ except for a finite number of \mathfrak{p} , and these form a multiplicative group. Though every element a of K can be written in the form

$$a = \prod_{\mathfrak{p}} \mathfrak{p}^{\alpha_{\mathfrak{p}}},$$

it does not follow conversely that every such combination of divisors corresponds to an element of K . A consequence of such a development is that in considering the arithmetic in K' of an algebraic extension of K , the usual definition of an algebraic integer as the root of an equation

$$\theta^n + a_1\theta^{n-1} + \dots + a_n = 0,$$

where a_1, \dots, a_n are integers in K follows as a theorem and not as a definition.

The properties of an algebraic function field K where K is a finite algebraic extension over $P = \Omega(t)$ where Ω is a complete field now follows. An account is given of the theory of divisor classes, the relation of divisors to ideals, the modul defined as the multiple of a given divisor, the differential divisors and the generalised Riemann-Roch theorem. There is also an account of "different" and discriminants.

The section concludes with accounts of quadratic fields, cyclotomic fields and class number considerations.

In reading this book it must be borne in mind, as stated in the preface, that the treatment of the subject presupposes a considerable knowledge of modern algebra such as can, however, be acquired from the author's booklet on Higher Algebra. Hence the reasoning is often of an abstract nature requiring careful consideration and close study, and elementary results are presented in such a form as to be sometimes not easily recognisable for what they are. It is no book for a beginner, the more so as it is much more of a handbook than a textbook; but the experienced reader will find it a source of much interest and pleasure.

Aesthetically the book is most satisfying. One realises that the notation is admirably selected, and I can think of no book on number-theory with a better choice. It is no mean feat to deal so harmoniously with the numerous symbols required for fields and subfields, moduls and rings, divisors and ideals, norms and discriminants. The reader will not be long immersed in the volume before he begins to feel and to recognise that the subject-matter is being handled by a master as it ought to be, and by an all-seeing expert who misses no possible aspect or illuminating illustration. Proofs are well presented and often are elegant, especially when compared with some given by other authors. The book is rich in ideas and the reader acquires a greater understanding of the theory. He will appreciate the real significance of those mysterious entities, the divisors which appear in so many shapes in number-theory, algebra, geometry and function-theory. One cannot help but realise the enormous amount of thought and work that have been necessary to bring some kind of finality into this book. How grateful must students in number-theory be to the author for the enormous amount of trouble he must have gone to in order to produce such a delectable treat. The reader will await with the greatest impatience Hasse's promised continuation of this book in a second volume.

L. J. MORDELL.

Transcendental Numbers. By C. L. SIEGEL. Pp. viii, 102. 16s. 1949. *Annals of Mathematics Studies*, 16. (Princeton University Press; Geoffrey Cumberlege, London)

There are not many topics in the domain of arithmetic and analysis called Diophantine approximation in which such epoch-making progress has been made in the last twenty-five years as in the study of transcendental numbers, that is, numbers x which are not roots of an equation of the form

$$a_0x^n + a_1x^{n-1} + \dots + a_n = 0,$$

where a_0, a_1, \dots, a_n are integers. First, there was Siegel's great memoir dealing with the transcendence of functions defined by linear differential equations, and in particular of the Bessel functions. Then there was the important work of Gelfond and Schneider on the transcendence of a^z for algebraic a and z , disposing of Hilbert's famous seventh problem. Finally there were the results of Siegel and Schneider on the transcendence of some elliptic integrals. These are all contributions of the greatest significance, and it is very desirable to have an easily accessible account of them and their relation to the classical work by Hermite and others on the transcendence of the exponential function. This has now been done in the present booklet, which has its origin in a course of lectures given by the author at Princeton.

It contains four sections. The first deals with the exponential function, e.g. questions on the irrationality of e , π , $\tan x$, the approximation to e^x by rational functions of x , the transcendence of e^a for real algebraic a ; and the more general result that if a_1, a_2, \dots, a_p are algebraic numbers such that

$g_1 a_1 + \dots + g_p a_p \neq 0$ for any rational integers g except $g_1 = 0, g_2 = 0, \dots, g_p = 0$, then the numbers $\exp a_1, \dots, \exp a_p$ are not connected by any algebraic equation with algebraic coefficients. The results are treated from a more general point of view than usual, and depend upon finding approximation forms, that is, polynomials P_1, P_2, \dots, P_m of assigned degrees in x such that

$$P_1 \exp(\rho_1 x) + \dots + P_m \exp(\rho_m x),$$

where the ρ 's are arbitrarily given constants, should have a zero at $x=0$ of the highest possible order. This is done by elementary function theory. The transcendence proofs are not so simple as some usually given, but they have the great advantage of containing ideas applicable to more general questions.

The root idea involved in proving the transcendence of one of a set of numbers $\xi_1, \xi_2, \dots, \xi_r$ is very simple. Assume that they are all algebraic. Then an algebraic integer ξ is constructed depending on a large parameter, say t , and such that $\xi \neq 0$, and so its norm satisfies $|N(\xi)| \geq 1$. The proof that $\xi \neq 0$ is usually the crucial point of the proof. Estimates are obtained for ξ and its conjugates, from which it can be shown that

$$|N(\xi)| \leq t^{-\delta}$$

for a positive δ independent of t . Then making $t \rightarrow \infty$, we have a contradiction since $N(\xi)$ is an integer not equal to zero.

The second section is the most abstruse in the book, and the proofs require a long chain of reasoning. It is concerned with transcendence of E functions, that is, those defined by a power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n / n!,$$

where the coefficients c_n belong to the same algebraic number field of finite degree over the rational number field, and where for any $\epsilon > 0$, the moduli of c_n and of its conjugates are $O(n^{\epsilon})$, and the least common rational denominator of c_0, c_1, \dots, c_n is also $O(n^{\epsilon})$. We have now again the question of finding polynomials P_1, P_2, \dots, P_m of assigned degree in x such that for m given E functions the expansion of $R = P_1 E_1 + \dots + P_m E_m$ starts with the highest possible power of x . Estimates for the coefficients of the P 's and then also for R follow from a lemma, most useful for such questions, which gives an estimate for an integer solution of a system of indeterminate linear equations with integer coefficients in terms of the magnitude of the coefficients. The E functions are then specialised as satisfying a system of homogeneous linear differential equations of the first order

$$\frac{dy_k}{dx} = \sum_{l=1}^m Q_{kl}(x) y_l \quad (k=1, 2, \dots, m),$$

where the coefficients Q_{kl} are rational functions of x . If for any solutions y_1, y_2, \dots, y_m and polynomials P_1, P_2, \dots, P_m

$$R = P_1 y_1 + \dots + P_m y_m$$

is not identically zero, we are led to the concept of normal systems for the E functions. For such we have the fundamental result: if the coefficients of Q_{kl} are algebraic numerical coefficients and all the power products for each $v = 1, 2, \dots$

$$E_1^{\alpha_1} \dots E_m^{\alpha_m}, \quad (\alpha_1 + \dots + \alpha_m \leq v),$$

form a normal system, then if $\alpha \neq 0$ is any algebraic number not a pole of the Q_{kl} , the m numbers $E_1(\alpha), \dots, E_m(\alpha)$ are not related by an algebraic equation with algebraic coefficients.

The generalised hypergeometric series leads to E functions of the types considered above, and the questions of normality can be settled for the Bessel differential equation. A number of theorems on the transcendence of Bessel's functions follow, and we need only cite one, namely, that $J_0(x)$ is transcendental for algebraic $x \neq 0$.

The third section contains the proof that a^b is transcendental for irrational algebraic b and algebraic $a \neq 0, 1$. Schneider's proof is given first since it is more closely related to the ideas developed in the second section for finding approximation forms. Then Gelfond's proof follows; priority for the proof of the theorem is due to him, although both proofs appeared in the same year 1934. In fact, the first results on this subject are due to Gelfond who proved in 1929 that a^b is transcendental for a quadratic imaginary irrationality b and algebraic $a \neq 0, 1$. Gelfond's proof, given in a form making it shorter than Schneider's, uses the obvious fact that the Taylor's series of an analytic function which is not a polynomial must contain infinitely many non-zero coefficients.

The last section deals with the transcendence of elliptic integrals of the first and second kinds. Some deductions from the results can be stated in a simple geometrical form. The arc of the ellipse $x^2/a^2 + y^2/b^2 = 1$ between $x = x_1$, $x = x_2$ is a transcendental number if a, b, x_1, x_2 are algebraic numbers unless the arc is zero. It seems to be a difficult matter to find results for elliptic integrals of the third kind, even also in the simple case of curves of genus zero. In fact it is not yet known if

$$3 \int_0^1 \frac{dx}{1+x^3} = \log 2 + \pi/\sqrt{3}$$

is an irrational number.

Professor Siegel has conferred a real service on pure mathematicians, at least on all those interested in analysis. I say so advisedly, since the subject-matter is such as to make the book an interesting one not only for narrow specialists, but also an eminently readable one even for those with only an elementary knowledge of analysis. It contains a number of most beautiful results, and makes one very conscious of the great possibilities of modern mathematics and of the power and skill of the mathematicians who advance it. The book is not only informative but most suggestive and stimulating, and gives one many occasions to think. It might well serve as a model of clear and concise exposition for other similar booklets. L. J. MORDELL.

Philosophy of Mathematics and Natural Science. By H. WEYL. Revised and augmented English edition based on a translation by O. HELMER. Pp. x, 311. 40s. 1950. (Princeton University Press; Geoffrey Cumberlege, London)

The advent of the two new theories of modern physics—relativity and quantum theory—shook to its very foundation a picture of the physical world which had gradually emerged during the preceding three centuries. The unsuspecting classical physicist was suddenly made to realise that at the basis of all his work there were hidden assumptions which were now contradicted by experimental facts. Newtonian mechanics itself, which had been so successful in describing the motion of molecules and celestial bodies, was recognised to be merely an approximation to the truth. It is not surprising that many mathematicians and physicists of this period began to examine the foundations of their subject.

Weyl is one of them. Not only did he live in this period, but he also actively contributed to both relativity and quantum theory. Few of the physicists who are alive to-day can join him in this privileged position: Einstein is one, Born and Pauli are others. In addition, Weyl was one of the first to give

systematic accounts of both relativity and quantum theory (*Raum Zeit und Materie*, 1918; *Gruppentheorie und Quantenmechanik*, 1928) in books which have remained standard accounts of their subject. It is therefore with great interest that English-speaking readers will welcome the present volume, which is a revised and augmented translation of his *Philosophie der Mathematik und Naturwissenschaft* which first appeared in 1927 in Oldenbourg's *Handbuch der Philosophie*. The titles of the different sections of the main text remain unaltered and there are six new appendices.

The first part of the book deals with mathematics in three chapters which are devoted to (a) mathematical logic, (b) number, continuum and the infinite, and (c) geometry. Though the subject-matter is now well known, Professor Weyl brings to his careful exposition of the fundamental methods of mathematical argument new and challenging illustrations. For instance, he introduces a type of truth table (p. 23, not present in the original edition) showing which of seven given points lie on seven given lines, from which general propositions ("through any two distinct points there goes exactly one line," etc.) can be verified. In this way the reader is introduced to intuitive and symbolical mathematics, the irrational and the infinitely small.

It will be of interest to all mathematicians, though not all will agree, that Weyl characterises the "life centre of mathematics" by saying that "mathematics is the science of the infinite" (p. 66). Recent broadcasts suggested that some *applied* mathematicians consider that it is our knowledge of the world around us, which has been the most fruitful source of mathematical development ("Mathematics is the language of science"). Again, Weyl considers that the following are the most fundamental mathematical facts (p. 84, not present in the original edition).

"(F_1) that the counting of a set of elements leads to the same number in whatever order one picks up its elements," and

"(F_2) that among the permutations of n (≥ 2) things one can distinguish the even and the odd ones". Yet elsewhere (p. 61) Weyl distinguishes between phenomenal knowledge and theoretical construction, delegating the latter to mathematics. It seems therefore that the fact F_1 may be expressed thus: "Consider any abstract set of elements closed with respect to a commutative and associative operation '+', it is possible to find linguistic examples of such a set in the world of phenomena". From this point of view, the following fact is possibly more fundamental: the ability of the human mind to abstract from individually exhibited objects arbitrarily selected qualities so that these objects become indistinguishable at this level of abstraction. It lies at the basis of counting and of language itself.

The chapter on geometry has been largely re-written. Here Weyl digs deeply into the foundations to find the "inner peculiarities" which distinguish the case $n=3$ among all n -dimensional spaces (p. 70), and to obtain a solution of "the riddle of left and right" (p. 84). His answer to the second problem is based on his fact F_2 . With regard to the former problem, Weyl observes (p. 136) that it is only in an odd-dimensional space that the extinction of a candle is followed by complete darkness about the candle (within a radius which increases with the speed of light). Evidently the problem of the dimensionality of space, as Weyl sees it, is to find a significant qualitative requirement which must hold for real space and which forces $n=3$.

The second part of the book deals with natural science in three further chapters. The author shows how the notion of space has emerged historically and psychologically, he analyses the way in which concepts and theories are formed, and concludes with an exposition of our present view of matter, fields, conservation theorems and causality. This account is factual and, although the analysis is penetrating, it brings little that is not by now familiar

to the English-speaking reader. There are, however, many references back to the less recent history of science (Kant, Leibniz, Cassirer, etc.), and this distinguishing feature makes this book much more than a critical guide to the literature (which the author modestly claims as one of the principal tasks of the book).

It is noteworthy that this volume does not show the wrinkles of old age, except indeed for one beauty wrinkle, which was left by the author's revising hand (p. 187). He refers to Heisenberg's decisive progress in quantum mechanics "which occurred while the MS. of this book was in preparation"—a footnote is added: "So written in 1926."

Shortly after the original edition was published, Weyl wrote his treatment of quantum mechanics to which reference has been made. An appendix on quantum mechanics was therefore to be expected in this revised edition. In a way it is a commentary and amplification of Eddington's remark that in quantum theory the observer is armed with a sieve. The properties of idempotent operators are here outlined and interpreted. When the author summarises the philosophically important features of quantum theory (p. 263), he omits, however, what is perhaps its most fundamental contribution, that physical quantities cannot *a priori* be treated as continuous variables. This point can now be elaborated with reference to recent attempts to construct a "quantised space-time" as background of our physical phenomena. Unfortunately there is no detailed account of the theory of measurement, and Weyl's remarks about causality seem to be rendered somewhat out of date by Born's recent brilliant discussion of the subject (*The Natural Philosophy of Cause and Chance*).

In the two appendices on chemical valency and on physics and biology Weyl explains his view that the laws of inheritance are ultimately based on quantum mechanics (p. 274), and he quotes and summarises recent speculations on the problem of life. The concluding appendix is a very readable account of what Professor Weyl regards as the main features of the physical world.

The book is well produced, has a good index and an attractive cover. With all quotations translated, it is, however, far from clear why two extracts from Brouwer have been retained in Dutch (pp. 61 and 63). Many readers will regard this as too severe a challenge!

There is a feature of Weyl's German style which has fortunately been preserved in occasional sentences: "More than anybody else has Hilbert, through the ingenious construction of suitable arithmetical models, contributed to the clarification of the logic relations . . ." (p. 22); "To Locke we are indebted for the classical distinction of 'secondary' and 'primary' qualities" (p. 111). The christening of this belated introduction of the subject as a "Weylism" the present reviewer finds it difficult to resist. Another quotation shall illustrate the suggestive power of Weyl's writing (p. 116):

"The objective world simply *is*, it does not *happen*. Only to the gaze of my consciousness, crawling upward along the life line of my body, does a section of this world come to life as a fleeting image in space which continuously changes in time."

P. T. LANDSBERG.

Sixth-Form Citizens. An inquiry of the Schools Committee of the Association for Education in Citizenship into the Social Content of Sixth-form Curricula. Pp. xvi, 287. 10s. 6d. 1950. (Geoffrey Cumberlege, Oxford University Press)

This is a book which every Sixth Form teacher of Mathematics should read, though it does not especially deal with his subject. This apparent paradox needs some justification!

Sixth Forms are rapidly increasing in size in all grammar schools. In the ten years 1938-48 the number of entrants for the Higher School Certificate rose from 13,202 to 26,322, a phenomenal advance of 98%. This is about 4% of the entire 18-year-old population. These boys and girls have among them the best brains of the country, and the Schools Committee of the Association for Education in Citizenship have been questioning whether the traditional curriculum provides them with a suitable preparation for life. No one queries the general excellence of the specialist training, but does the student in Arts or Science get sufficient "social education", as it is often called, so that he can take his proper place as a member of a large community?

Much has been written about moral education. Do schools really put the development of the virtues in the forefront of their programme? The extent of juvenile delinquency makes many people uneasy. It is vital that the best brains, or the Sixth Formers, should have a clear idea of what they owe to the country. We do not want to produce a discontented class, who demand their rights and neglect their duties. No one would dispute this, but the method of attainment varies. Many schools would claim that they are indirectly inculcating the virtues in all their activities, but in this vital matter it is good that a report should be published where the experiences and methods of many schools can be pooled for the information of all.

In general perhaps the objective laid down by the Ministry of Education, and very commonly followed, has been to allow the pupil to specialise in his own subjects to the extent of not more than three-quarters and not less than two-thirds of the school time. The remainder should be used for additional subjects to provide "balance". For a mathematical student these latter subjects would probably be English, a language and Religious Knowledge. But it is interesting to know, by the result of questionnaires, that in many schools these "balance" subjects which should amount to 9-11 periods in a 35-period week are often greatly curtailed. Further, it is even more interesting that this curtailment is generally most severe in the case of an Arts student. Very often this is due, in both cases, to the demands of specialist teachers who are out for good results in examinations, and especially for University scholarships. No one can blame them as long as the public regard such successes as the hallmark of excellence. Hence competition becomes severe, and the standard of scholarship papers rises, providing the teacher with an excuse for a further diminution of all non-specialist subjects.

This, of course, is an old complaint, and the reader must ask how social studies can possibly be added to the overcrowded curriculum. Yet no firm answer is given in the book, though suggestions are made. The subjects concerned may be called Economics, Statistics, Civics, Current Affairs, and there are plenty of other names. Indeed, Politics, Philosophy and Logic as well have their supporters. The Committee consider, too, whether the approach be direct, or indirect through the medium of history or modern geography. Both ways are possible, and are dealt with in separate chapters, but Mathematics and Science scarcely lend themselves to this indirect approach.

The Committee are firmly convinced that the position in which Britain and the world find themselves to-day make the inclusion of social content studies in the Sixth Form imperative, and that large numbers of our most intellectual pupils fail to grasp the true principles of democratic citizenship. They also state that time is being found for these studies in a number of schools, which are, at the same time, most successful in gaining University Scholarships, and they do ask that, in some small way, the curriculum should be recast.

Under these circumstances, even remembering the timetable difficulties,

the book ought to be read, as it is a fascinating study by a body of experts, and will impress and stimulate the reader. We could wish it was better written; obscurities and long words are not really necessary in educational writing. But the substance is most important, and we strongly recommend it even to those who are fully satisfied with the work of our Sixth Forms, and we feel that even the most experienced teacher will gain by its perusal.

We should add that Mathematics, as such, has little place in this volume. Yet the value of statistical probability is admitted as an essential piece of equipment for understanding the modern world. The author states that either one is at the mercy of statistics or one is a master of them. Sir Cyril Burt, in his "Transfer of Training", a most interesting essay printed in the second Part, adds: "any conclusions on social matters must be based on arguments from probabilities, not from certainties". Here is an argument for its inclusion!

Finally, we cannot resist quoting the following statement. "Although not directly related to the main enquiry it is surely desirable that young teachers should have gained a sound knowledge of Astronomy before they embark upon their careers." There are many of us who would fully agree with that!

W. F. BUSHELL.

The Variational Principles of Mechanics. By CORNELIUS LANCZOS. Pp. xii, 307. 42s. 1949. (Toronto: University Press; London: Geoffrey Cumberlege)

Despite the author's modest claim that "the present book is conceived in a humble spirit and is written for humble people", the reviewer believes that this is one of the most important works on the philosophy of mechanics which has appeared since Mach's classic treatise. The present book, which is the fourth in the renowned Toronto Series, is essentially mathematical, both in actual treatment and in spirit, but it is this mathematical orientation which provides the necessary complementary perspective to that of the great Viennese positivist. In Mach's philosophy mathematics is merely a species of shorthand language for expressing complicated relations tersely. Consequently, it is not surprising that he had no true appreciation of analytical mechanics, and gave it as his opinion that "No fundamental light can be expected from this branch of mechanics. On the contrary, the discovery of matters of principle must be substantially completed before we can think of framing analytical mechanics the sole aim of which is a perfect *practical* mastery of problems." As Dr. Lanczos points out: "According to this philosophy, the variational principles of mechanics are not more than alternative mathematical formulations of the fundamental laws of Newton without any primary importance." It is the prime object of the present book to expose the inadequacy of this point of view.

The author begins by reminding us that from the time of Newton and Leibniz the science of mechanics has developed along two main lines, one originating in the *vectorial* concepts of "force" and "momentum", the other in the *scalar* concepts of "kinetic energy" (*vis viva*) and the "work function". Everyone knows that Newton shares with Leibniz the honour of discovering the differential calculus, but I expect that to many it will come as somewhat of a surprise to learn that in fact Leibniz was also the originator of the second main line of mechanics, "usually called 'analytical mechanics', which bases the entire study of equilibrium and motion on two fundamental scalar quantities, the 'kinetic energy' and the 'work function', the latter frequently replaceable by the 'potential energy'". From the mathematician's point of view, in his mechanics as well as in his differential calculus, Leibniz's methods were destined to prove more powerful in the long run than those of

the Great Master himself. In the contrast between the respective theories of dynamics of Newton and Leibniz we see the ultimate origin of the notorious dilemma which confronts the contemporary student of (fundamental) mathematical physics—the dilemma of particle *versus* field.

It is well known that in the case of particles whose motion is “unconstrained” the Newtonian and the variational methods give essentially the same results. For constrained systems, however, the Newtonian method can be applied only if definite hypotheses, *e.g.* the third law of motion, are invoked. The variational method avoids this appeal, but in so doing it appears to be less general than the other, for dissipative forces seem to be beyond its scope. This traditional distinction between the applicability of the two methods may, however, be purely a consequence of our particular way of looking at specific dynamical problems. Frictional forces originate from inter-molecular phenomena, and as the author remarks: “If the macroscopic parameters of a mechanical system are completed by the addition of microscopic parameters, forces not derivable from a work function would in all probability not occur.” Significantly, in problems of modern quantum mechanics we still look for the appropriate Hamiltonian.

To the mathematician the peculiar appeal of the Leibniz-Euler-Lagrange-Hamilton technique is due to the complete freedom it allows for our choice of coordinates. Indeed, the variational approach to mechanics anticipated the modern ideas of invariance and covariance on which the Principle of Relativity is based. Whereas the Newtonian equations of motion had to be drastically modified to accord with this principle, the variational equation remained valid, except for the requirement that the “action” had to be chosen so as to be invariant under any coordinate transformation. Thus, in marked contrast to Mach, the present author argues that: “In the light of the discoveries of relativity, the variational foundation of mechanics deserves more than purely formalistic appraisal,” and “the application of the calculus of variations to the laws of nature assumes more than accidental significance”.

The author is careful to point out that his treatise does not attempt to compete with the standard textbooks of advanced mechanics, but rather to supplement them. In acknowledging his principal sources of reference, he mentions first and foremost Whittaker's *Analytical Dynamics*, and in the opinion of the reviewer Dr. Lanczos' book should be regarded as a companion volume to that mathematical classic. It comprises the following chapters: (1) The Basic Concepts of Analytical Mechanics; (2) The Calculus of Variations; (3) The Principle of Virtual Work; (4) D'Alembert's Principle; (5) The Lagrangian Equations of Motion; (6) The Canonical Equations of Motion; (7) Canonical Transformations; and (8) The Partial Differential Equation of Hamilton-Jacobi; preceded by a stimulating philosophical introduction and rounded off by a useful concise historical survey, together with a brief list of books recommended for collateral reading. Each chapter, except the last, is divided into sections, the number per chapter ranging from six to fifteen, and each section concludes with a neat summary displayed in box form so that the essential points can be quickly recaptured. Although the author's emphasis is naturally on the variational approach to mechanics, the vector technique is used when the situation demands it, *e.g.* for moving axes. Problems are scattered throughout the text; they are, in the main, of a simple character and have been chosen so as to exhibit the general principles discussed. The total number of problems, however, is not large, but this is understandable, as the book is not intended to replace existing textbooks. Among the careful discussions of physical principles which lie behind specific mathematical techniques, one which particularly appealed to the reviewer

was the discussion in Chapter III of the physical interpretation of the Lagrangian multiplier method; but indeed the whole of this chapter, on a subject which is notoriously difficult to teach, is both penetrating and stimulating.

One feature of Dr. Lanczos' book to be particularly commended is his inclusion of topics such as Liouville's theorem which are usually neglected by writers on dynamics, as distinct from statistical mechanics. Again, in his chapter on the Hamilton-Jacobi equation he links up the classical technique with the quantum conditions of Sommerfeld and Wilson by applying Delaunay's method of treatment of separable periodic systems, and also refers to the theories of de Broglie and Schrödinger. Since, in recent years, analytical dynamics has come to be regarded by many as a somewhat outmoded, although not quite dispensable, part of the young mathematician's training, the author's attempt to place the subject in its correct perspective is wholly admirable. As he is at pains to explain, the two great achievements of contemporary physics, the theory of relativity and the quantum theory, are both closely allied to analytical mechanics: "In spite of the radical departure of the new concepts from those of the older physics, the basic feature of the differential equations of wave-mechanics is their *self-adjoint* character, which means that they are derivable from a variational principle. Hence, in spite of all differences in the interpretation, the variational principles of mechanics continue to hold their ground in the description of all the phenomena of nature."

The one important objection to be brought against this book by those of us who live on this side of the Atlantic is its high price, which will probably place it beyond the reach of many who would otherwise wish to buy it. It ought, however, to be acquired by every university and mathematical departmental library. The production is up to the high standard which we have come to expect of the Toronto Series.

G. J. WHITROW.

Tensor Calculus. By J. L. SYNGE and A. SCHILD. Pp. xi, 324. 45s. 1950. (Toronto: University Press; London: Geoffrey Cumberlege)

This book fills a niche that has long been waiting to receive it. Despite the growing importance of the tensor technique for so many branches of mathematical physics, there are remarkably few books which can be wholeheartedly recommended to a student who is not primarily a geometer. McConnell's *Applications of the Absolute Differential Calculus* was an elegant pioneer textbook. Brillouin's well-known book is, of course, in a foreign tongue, and includes much else besides the purely mathematical technique. Consequently, without any intention to disparage these older works, the reviewer believes that, but for the calamitous effect of devaluation, this latest volume in the Toronto Series would be the most suitable book to recommend to the student who requires a good working knowledge of the tensor technique rather than an introduction to modern differential geometry. We must hope that it will be made accessible to as many as possible through libraries.

The authors' stated intention is to provide a general introduction to the subject without being exhaustive and without discussing the historical origin of the technique. Although the tensor calculus was invented by Ricci and Levi-Civita at the end of the last century, it was regarded by most mathematicians as a barren formalism until, in the second decade of this century, Einstein discovered that it was the mathematical tool which he needed to generalise his theory of relativity to all frames in all conceivable types of motion and to all systems of coordinates. The usefulness of the tensor calculus, however, is by no means restricted to the General Theory of Relativity. The authors claim that it comes near to being a universal language in mathematical physics, since it not only provides a compact form of expression for

general equations, but "it also guides us in the selection of physical laws by indicating automatically invariance with respect to the transformation of coordinates". Since most expositions of the subject are written primarily for students of relativity, the authors have decided to exclude relativity almost completely from their account, and instead have mainly concentrated on the applications to classical mathematical physics.

The book consists of eight chapters, two appendices and a short bibliography and index. The first four chapters and the last two concern the pure technique, whereas the fifth and sixth chapters are devoted to some of its applications. Chapter One begins from first principles with the idea of generalised space, the summation convention, contravariant and covariant vectors and tensors. In the following chapter, the authors introduce the concept of Riemannian space by easy stages, first discussing curvilinear coordinates in Euclidean space. (In the first exercise on p. 31, read a_{11} and a_{22} for a^{11} and a^{22} in the denominators.) Geodesics are introduced, and their differential equations are derived by an argument which is valid irrespective of whether the metric form is definite or indefinite. (In the pioneer investigations last century a definite form was usually assumed, but with the advent of Minkowskian space-time the need for indefinite forms to be included became apparent.) The core of this chapter consists of a careful discussion of the differentiation of tensors, introducing the usual Christoffel symbols. The authors remark, however, that these symbols are often clumsy to handle in explicit calculations, and they show how in many instances it is not necessary to use them. The chapter concludes with a long section on special coordinate systems and a short one on the Frenet formulae. While stressing that the "democratic principle" of avoiding reference to particular coordinate systems is "the idea underlying the whole subject", and while recognising that in a general Riemannian space there exists no system of coordinates as simple as rectangular Cartesians, the authors draw attention to several systems with simplifying properties. In introducing *local Cartesians*, the subtleties associated with the reduction of a quadratic form to a sum of squares are relegated to an appendix.

The idea of curvature is introduced in the third chapter, the fourth being devoted to a discussion of spaces of constant curvature. From the start, the authors emphasise that curvature is not to be regarded as something measured by comparison of the space in question with another space, in particular a Euclidean space, but must be understood as something intrinsic to the original Riemannian space itself. In a highly interesting section, the geometrical significance of the sign of the curvature is discussed, and the behaviour of geodesics diverging from a common origin in the space is shown to depend on this sign. The subsequent discussion of parallel transport is vivid and illuminating. Cartesian tensors are introduced in the chapter on spaces of constant curvature; but the distinction between topologically different spaces of this type, although mentioned, is not discussed at this stage.

Nearly one hundred pages, about a third of the whole book, are devoted to the applications of the tensor calculus to classical mathematical physics, including the dynamics of particles and rigid bodies, rotating frames of reference, general dynamical systems and their topological properties, hydrodynamics, elasticity, electromagnetic radiation and Maxwell's equations in four-dimensional notation.

The final chapters cover relative tensors, ideas of volume, Green-Stokes theorems and non-Riemannian spaces. The three former topics are introduced in some detail, but the latter is discussed very summarily as the authors object is merely "to give the reader an idea of some of the more modern developments". Nevertheless, the authors indicate how, beginning with

spaces of symmetric connection, Weyl spaces and Riemannian spaces, respectively, can be obtained by successive specialisation.

Each chapter concludes with a summary of formulae. An outstanding feature of the book is the wealth of examples contained in it. The general production is excellent.

G. J. WHITROW.

Classical Mechanics. By HERBERT GOLDSTEIN. Pp. xii, 399. \$6.50. 1950. (Addison-Wesley Press Inc., Cambridge 42, Mass.)

The author's aim in this book was to develop an account of advanced classical mechanics which by internal comment, emphasis, and illustrative example should blaze the trail to modern theoretical physics; Dr. Herbert Goldstein of Harvard University is to be congratulated on a very stimulating and successful achievement of his objective.

There is, of course, no unique path from the old to the new and, as time goes on, other books will be written with the same object in view, but with a different choice of subject-matter. I feel, however, that Dr. Goldstein's work will stand the test of time for many years to come and will take its place by the side of acknowledged masters such as E. T. Whittaker's *Analytical Dynamics*, Sommerfeld's *Vorlesungen über Theoretische Physik*, Margenau and Murphy, *The Mathematics of Physics and Chemistry*; Jeffreys and Jeffreys, *Methods of Mathematical Physics*, etc.

The book is written for American graduate students who research in, or propose to research in, branches of modern theoretical physics, and the text makes it clear how extensive a knowledge of the techniques of advanced mathematics is required by present-day investigators in this field—but, lest an unsophisticated reader might be led to believe that all American students of mathematical physics can take this book in their stride, there is a revealing aside that "the inadequately prepared graduate student" is "an all-too-frequent occurrence".

In the field of Physics the previous knowledge assumed by this book is "at least a term's exposure to modern physics" and "an intermediate course in mechanics"—but I do not know what this means. I note, however, that for some sections an elementary acquaintance with Maxwell's equations and their simpler consequences is required. The mathematical background which is assumed is that of "the customary undergraduate courses in advanced calculus and vector analysis"—which, from internal evidence, I interpret as equivalent to the standard of mathematical knowledge shown by a good student of mathematics just about to enter the final Honours year in an English University School of Mathematics. The more complicated mathematical techniques are developed as they are needed, and an effort has been made to keep the book more or less self-contained.

The book has many virtues, and among the first of these I would single out the feeling of vitality and the sense of clarity which the author has infused into his explanations. Next, the choice of subject-matter is interesting and instructive. After a compact, elegant survey of elementary principles of mechanics, there is an account of variational principles and Lagrange's equations. The two-body central force problem is discussed and is followed by an account of the kinematics of the motion of rigid bodies. The motion of such bodies is then treated in some detail, and a review of special relativity in classical mechanics succeeds it. The Hamiltonian equations of motions, canonical transformations, the Hamilton-Jacobi theory, small oscillations, and an introduction to Lagrangian and Hamiltonian formulations for continuous systems and fields then form the subject-matter of consecutive chapters.

The section on central forces includes the kinematics of scattering and the

classical solution of the problem of scattering. The rotations of a rigid body are treated from the standpoint of matrix transformations, and Euler's theorem on the motion of a rigid body is presented in terms of the eigenvalue problem for an orthogonal matrix. Classical mechanics has in the past laid emphasis on static forces dependent on position only, such as gravitational forces, but in the present book forces and potentials which are functions of velocity are introduced from the outset; the importance of the velocity-dependent electro-magnetic force which is frequently encountered in modern physics is the reason for this.

A detailed bibliography, selected references for further reading—coupled with comments which are sometimes refreshingly acid, a glossary of important symbols with textual cross-references, a fairly full index, excellent printing, excellent diagrams, especially some of those which refer to three-dimensional configurations, small selections of well-chosen examples, all contribute to make this an interesting, valuable and notable book.

L. R.

The Mathematical Theory of Huygens' Principle. By B. B. BAKER and E. T. COPSON. Second edition. Pp. vi, 192. 21s. 1950. (Geoffrey Cumberlege, Oxford University Press)

Baker and Copson's monograph on the mathematical theory of Huygens' principle in optics and its application to the theory of diffraction is well known to those who conduct advanced lecture courses or seminars on the partial differential equations of mathematical physics. In this second edition the four chapters of the original are virtually unchanged, except for the addition of references to more recent work and the correction of minor misprints and errors. The main change is the addition of a chapter on the application of the theory of integral equations to problems of diffraction by a plane screen. In addition to a discussion of the classic papers of Rayleigh and Schwarzschild, there is given an account of recent work in the subject by Professor Copson, E. N. Fox and others. The chapter ends with a lucid exposition of the variational principle of Levine and Schwinger which has been of value in the solution of problems in short wave radio in which no rigorous solutions are known. There have been many interesting developments in the theory of diffraction since 1939, both in the theory of sound and in radar; the account given here is an excellent introduction to these studies, and will be of great value to the student beginning a study of the recent literature. This second edition, like the first, is characterised by the lucidity of the authors' style and the excellence of production traditionally associated with the Clarendon Press.

I. N. SNEDDON.

Die Mathematischen Hilfsmittel des Physikers. By ERWIN MADELUNG. Fourth edition. Pp. xx, 531. DM. 47; geb. DM. 49.70. 1950. (Springer, Berlin).

"Madelung" needs no introduction to theoretical physicists or applied mathematicians. It is to be found, next to "Jahnke-Emde", on their shelves already. A new edition is rendered necessary only by the fact that the amount of mathematical knowledge which forms the working equipment of the theoretical physicist is growing fast. Some idea of the rapidity of this growth may be gathered from the fact that whereas the second edition published some twenty-five years ago, was a slim volume of some 280 pages, the fourth edition is nearly twice as large. It would seem that the days when a knowledge of the "special" functions of mathematical physics sufficed are gone, and, as the result, the special value of "Madelung" is that, unlike other compendia of this kind, it takes as its province not merely those topics which are thought of as "advanced calculus", but the whole of mathematics,

pure and applied. Thus, in this fourth edition, the reader will find full accounts of the theory of groups, tensor calculus, matrices and transformation theory, as well as of the more familiar topics in calculus and differential equations. In the field of applied mathematics, quantum theory, relativity, thermodynamics and statistical mechanics are discussed as fully as the classical subjects, mechanics and electricity. The accounts of them are so good that it is a matter of regret that the quantum theory of radiation is not treated as fully. The main defect of the book is its lack of references. A work of this kind must necessarily state results more briefly than meets with the approval of the pure mathematician, but it should compensate for this defect by referring the reader to the sources from which they are taken. The book is attractively printed and set out, but its paper covers are a serious drawback to a work which will be used frequently.

I. N. SNEDDON.

Die Entwicklung der Infinitesimalrechnung. I. By OTTO TOEPLITZ. Pp. 180, ix. DM. 19.60 : geb. DM. 22.60. 1949. Die Grundhehren der mathematischen Wissenschaft, 56. (Springer, Berlin)

The history of mathematics offers many examples of the recurring cycles to which Spengler drew attention in his comparative morphology of civilisations; the extravagant carefree expansion in the initial stages of a development, the onset of critical doubts followed by codification of the completed structure in a rigid formalism, and a brief twilight return of the original creative urge. To what extent, however, does a student, living on the threshold of this final flowering of our mathematic, making his first contact with the language of a fully-grown discipline like the Calculus, find confidence or assistance in a survey of the forms out of which his present study evolved? Toeplitz answers this question by saying that, since the history of mathematics is the history both of what is best and what is worst in mathematics history as such can be of little help to the beginner, but a series of selections from history of its highest achievements, a series adequate at any rate for a logical reconstruction of the past, such a series truly helps the beginner by resolving his bewilderment about how, and in what way, and why, just these particular forms are valuable and useful. This process of selection from history is what Toeplitz calls the *genetic* method, and it is the purpose of this introduction to the calculus to show the genetic method in action. The result is a stimulating and enlightening book which invites comment from several standpoints.

Regarded as a teaching text for use with the science specialist in his last year at school, this first volume covers much of the ground of the standard school courses in the Calculus and Newtonian attractions. There are chapters on the definite integral, gradients, maxima and minima, differentiation of the circular and logarithmic functions and a rather fuller account than usual of Newton's derivation of the law of gravitation from Kepler's three propositions. From this standpoint alone the book will be welcomed by teachers seeking a trustworthy statement of the calculus in a novel guise.

The formal development of the book follows orthodox lines. The number system is taken to be the class of endless decimals, and the convergence of a bounded monotone sequence (proved by the method of repeated subdivision) is the key foundation theorem. The definition of the definite integral, however, is of a constructional type; sequences of upper and lower sums S_{2n} , T_{2n} are defined by repeated bisection, and these sequences are shown to have a common limit for a stretchwise monotonic function. The integral is then set free from the special sequence of subdivisions on which it was based, and the convergence of the sequence of upper or lower sums over any succession of subdivisions of zero norm is established. It would seem preferable to define the

integral as the common limit (when it exists) of upper and lower sums over a sequence of subdivisions of zero norm (following the usual practice) and then prove the existence of the limit for stretchwise monotonic functions rather than formulate the *definition* in terms of monotone functions. Toeplitz's object, of course, is to present the integral as the limit of some one *definite* sequence; the danger is that the beginner will not realise that the limits of the special sequences S_{2n} , T_{2n} may exist (and be equal) for functions which are not monotone, and so apparently the integral may exist, and yet fail to be the limit of S_n for a general sequence of subdivisions of zero norm.

Although Toeplitz disclaims any intention of writing a history of mathematics, yet it is as a history that this introduction to the Calculus makes the widest appeal. The historical illustrations are so aptly chosen and the judgement so balanced and penetrating. Where would we find a better account of the critical steps in Napier's discovery of the logarithmic function, or of Kepler's wonderful mathematical gifts?

Toeplitz shows great acumen in his analysis of the Newton-Leibniz controversy. He distinguishes three stages in the development of the Calculus; the calculation of tangent slopes and maxima and minima, the evaluation of areas, and the fundamental formula equating the rate of change of an area to the ordinate of its bounding curve. This last he holds to be the decisive discovery, and since Barrow published it in his "*lectiones geometricae*" in 1670 neither Leibniz nor Newton can be said to have *created* the Calculus. (Would it not, in fact, be absurd to credit either Newton or Leibniz with the discovery of a technique of which Kepler was so great a master half a century earlier?) This conclusion in no way detracts from the greatness of Newton or Leibniz's achievements. Leibniz brought to perfection in a single stroke the pregnant symbolism which gave the Calculus its extraordinary power in the eighteenth century, whilst Newton forged the theory of infinite series and made of the Calculus a *Theory of Functions*.

Toeplitz's *Infinitesimalrechnung* is a posthumous publication; the author died in Jerusalem in 1940 exhausted by his experiences in Germany before his emigration the previous year. The manuscript was prepared for the press by Professor Röhle. Although the present publication is described as volume one, the editor seems to be of the opinion that the material remaining in his hands will probably not suffice for a second volume. R. L. GOODSTEIN.

Grundzüge der Tensorrechnung in analytischer Darstellung. II Teil: Tensoranalysis. By A. DUSCHEK and A. HOCHRAINER. Pp. vii, 338. 42s. 6d. 1950. (Springer, Vienna)

The absolute differential calculus, or tensor calculus as it is now commonly called, invented at the end of the last century by Ricci and Levi-Civita, gained little attention until it proved in 1916 to be the mathematical instrument appropriate to the formulation of general relativity. Thereupon it became a subject of wide and intensive research, and for some fifteen or twenty years occupied a front place on the mathematical stage. Several things combined to keep interest in it alive during the 1920's: first, the lack of a satisfactory unified field-theory of gravitation and electromagnetism remained an outstanding challenge to mathematical physicists; secondly, the publication in 1928 of Dirac's equations for the electron, which at first seemed to lack complete accord with the relativistic principle of covariance, led to the invention of spinors (spin-tensors) and to a subsequent generalization of the idea of a tensor; and thirdly, the new theories were bound up with wide developments in geometry, and with new concepts of space, of intense interest to pure mathematicians. By the early 1930's, however, interest in these theories had begun to wane, and, although research in them continued, they ceased to

occupy their former leading position in mathematical thought. In that respect they were typical of most mathematical theories. When, for a decade or two, such a theory has been in the forefront of research, it begins to recede and gradually becomes "classical". During its period of greatest activity, an accumulation of detail on the one hand, and a growing generalization and abstraction on the other, lead to a situation where newcomers into the field despair of catching up, and existing workers begin to feel either that the subject is largely worked out, or that it has become too abstract and needs to be "brought down to earth"—that is, to return to a stage beyond which they have themselves progressed. Consequently the centre of interest tends to shift, sometimes to cognate branches and sometimes to wholly new theories.

It is in this sense that the tensor calculus has become classical. Even while it was receding, however, the seeds of its revival were being sown. Its main mathematical application had been to differential geometry in the small, but a few writers had kept in view questions of differential geometry in the large, the whole being placed on an axiomatic basis by Veblen and Whitehead in their Cambridge Tract (No. 29), *The Foundations of Differential Geometry* (1932). Little headway was made then, however, in the development of differential geometry in the large, and relatively little has been made since; but interest in it is perceptively reviving, and there seems little doubt that it will become a main field of research during the next ten or fifteen years. For workers in this new field of investigation some knowledge of the tensor calculus will be essential, but much of the detail and elaboration of the immediate post-relativity period will be dispensable.

While the tensor calculus itself was undergoing development during the 1920's, hand in hand with the new physical and geometrical theories, it was being applied to branches of mathematics and of mathematical physics already classical. Representative of these developments were two books, both published in this country in 1931, namely, McConnell's *Applications of the Absolute Differential Calculus* and Harold Jeffreys's *Cartesian Tensors*, which, though differing in emphasis, purpose and style, both gave accounts of the application of the tensor calculus to elementary geometry and to classical physics, including dynamics. During the same period the notation and techniques of the tensor calculus proved to be of value in connection with linear partial differential equations of the second order and with the theory of continuous groups. The tensor calculus is thus one of the most powerful instruments of modern mathematics, so wide in its "pure", "applied" and technological applications that every advanced student of the mathematical sciences should have some acquaintance with it. Well-written introductory works like the one under review are therefore to be welcomed.

The author of any introductory book on a subject of such wide ramifications must necessarily be selective, since comprehensiveness is hardly possible. Furthermore, he must form a clear idea of the class of readers for whom he is writing, lest he fall between two stools and produce a work, of a type by no means uncommon, which is too difficult for the beginner and too elementary for the expert, or else too "pure" for the physicist and too "applied" for the purer type of pure mathematician. The present authors make no mistake of that kind. They have frankly in view the needs of the technologist, while bearing in mind those of the physicist and mathematician.

The work is planned in three parts. Part I, not here under review, published in 1946 (second edition 1948), was concerned with the algebra of three-dimensional (rectangular) cartesian tensors. The second part is devoted to tensor analysis (field theory and differential geometry), and the third, which has not yet appeared, is to contain technical applications.

Part II opens with an account of classical differential geometry, covering

In that decade to recede activity, and the field that the needs and which tends theories. Even its in the geometry and differential development made doubt in years, tensor the im- ing the , it was already mathematically and linguistically. It may be hoped, however, that further use will be made in the third volume of general (non-cartesian) tensor analysis, since the brief account of it in Part II, occupying as it does only about a sixth of the total number of pages of the volume, might otherwise seem hardly worth while even as an introduction. But apart from that possible criticism (as well as the very minor one that the authors adhere to the now outdated notation $\{\overset{\circ}{g}_{ij}\}$ for the Christoffel symbols, instead of $\{\overset{k}{g}_{ij}\}$), the work is to be warmly recommended, to mathematicians as well as to physicists and technologists. An English-speaking beginner, unfamiliar with the English equivalents of German technical terms, might in reading it experience the usual difficulties of those who attempt to learn a new subject in a foreign language, but would find it a useful supplement to works like those of McConnell and Jeffreys or to the more recent volume of Synge and Schild. The printing is beautiful, the paper of excellent quality, and the binding strong enough to withstand all normal use.

H. S. RUSE.

Calculating Instruments and Machines.* By D. R. HARTREE. Pp. ix, 138. 21s. 1950. (Univ. of Illinois Press; Cambridge University Press)

Professor Hartree is an acknowledged authority on the subject of this book on both sides of the Atlantic. The present publication is based on his lectures given at the University of Illinois in the second half of 1948; such an origin disarms all criticism as to contents, since an introduction to the general prin-

* In Professor Hartree's article, "Automatic calculating machines", *Mathematical Gazette*, December 1950, the item No. 10 in the list of references (p. 252) gives the present volume as published by the University of Illinois Press. To avoid confusion, it may be pointed out that this was the American edition; the English edition was published in 1950 by the Cambridge Press, as here described.

ciples as they have developed from Babbage's and Kelvin's ideas, and a great number of illustrations supplied from the experience of the author, is clearly an admirable plan for students without specialised knowledge in this field. Presumably the same description should apply to the reader to whom this book is primarily addressed, but he cannot be but disappointed at finding the author interrupting his exposition, where no obvious limitation of time or space appears to force him to do so. Thus, while all readers will find something interesting and stimulating in every section, it is hard to imagine anyone who would not ask for more.

As explained in Sections 4.1 and again in 5.2, the book is concerned with the functional rather than with the structural aspect of calculating tools. The first chapters deal with instruments, this term being nearly, but not quite, synonymous with "analogue machines". Their main topic, the Differential Analyser, has been described by Hartree himself (see the *Gazette*, vol. 22, p. 342) and in other publications (J. Crank's book has been reviewed in the *Gazette*, October 1948). Various methods of its applications to partial differential equations are due to Hartree and his school in England, and Chapter 3 of his book is devoted to them. Many new methods will no doubt be developed in future for work with the Differential Analyser, but one feels that the time for pioneers has passed.

The same can certainly not be said of digital machines. Here the pure mathematician as well as the engineer has ample scope. In the first place, the philosophy of the machine, concerned with the logical equivalence of arithmetical operations and instructions, with programming and coding, with economy of preparation and performance, is new and often still contentious. Then the transformation of ideas into hardware is a challenge to be met by modern methods only. After all, it was due to the technical shortcomings of their times that Babbage's ideas lay fallow until recently. Moreover, the existence of calculating tools compels the mathematician to think of his problems in a new way, well illustrated on many pages of Hartree's book. In Chapter 5 we are introduced to such notions as serial and parallel operations, static and dynamic storage and n -address codes. Chapter 8, perhaps the most interesting of all, is rather misleadingly called "Projects and Prospects". It contains an account of delay line storage, a description of the logical basis (Boolean algebra) of the machine, and hints concerning programming. The last chapter deals with numerical analysis as carried out by high-speed automatic digital machines. As a matter of fact, much that Hartree says on this point applies equally well to computation without such sophisticated equipment.

The reviewer cannot conclude without asking for a textbook on high speed machines, which would cater for the needs of those who design them, of those who use them, and last but not least of those who wish to put their problems into a form which is amenable to treatment with modern tools. It would also be extremely interesting to know Professor Hartree's ideas on how to organise the service, which the machines, once they exist in sufficient numbers (how many?), can give to the mathematical, engineering and industrial public.

Print, paper and graphs are satisfactory, and the reviewer has found very few misprints (p. 102, l. 12, the first reference should be to Fig. 55 (c)).

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